

CROSSED PRODUCTS BY ENDOMORPHISMS OF  $C_0(X)$ -ALGEBRAS

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ABSTRACT. In the first part of the paper, we develop a theory of crossed products of a  $C^*$ -algebra  $A$  by an arbitrary (not necessarily extendible) endomorphism  $\alpha : A \rightarrow A$ . We consider relative crossed products  $C^*(A, \alpha; J)$  where  $J$  is an ideal in  $A$ , and describe up to Morita-Rieffel equivalence all gauge-invariant ideals in  $C^*(A, \alpha; J)$  and give six term exact sequences determining their  $K$ -theory. We also obtain certain criteria implying that all ideals in  $C^*(A, \alpha; J)$  are gauge-invariant, and that  $C^*(A, \alpha; J)$  is purely infinite.

In the second part, we consider a situation where  $A$  is a  $C_0(X)$ -algebra and  $\alpha$  is such that  $\alpha(fa) = \Phi(f)\alpha(a)$ ,  $a \in A$ ,  $f \in C_0(X)$  where  $\Phi$  is an endomorphism of  $C_0(X)$ . Pictorially speaking,  $\alpha$  is a mixture of a topological dynamical system  $(X, \varphi)$  dual to  $(C_0(X), \Phi)$  and a continuous field of homomorphisms  $\alpha_x$  between the fibers  $A(x)$ ,  $x \in X$ , of the corresponding  $C^*$ -bundle.

For systems described above, we establish efficient conditions for the uniqueness property, gauge-invariance of all ideals, and pure infiniteness of  $C^*(A, \alpha; J)$ . We apply these results to the case when  $X = \text{Prim}(A)$  is a Hausdorff space. In particular, if the associated  $C^*$ -bundle is trivial, we obtain formulas for  $K$ -groups of all ideals in  $C^*(A, \alpha; J)$ . In this way, we constitute a large class of crossed products whose ideal structure and  $K$ -theory is completely described in terms of  $(X, \varphi, \{\alpha_x\}_{x \in X}; Y)$  where  $Y$  is a closed subset of  $X$ .

## INTRODUCTION.

Crossed products by endomorphisms proved to be one of the major model examples in classification of simple  $C^*$ -algebras. The first instances of such crossed products, informally introduced in [9], were Cuntz algebras  $\mathcal{O}_n$ . Rørdam [49] and Rørdam and Elliott [12] established the range of  $K$ -theoretical invariant for all Kirchberg algebras by showing that crossed products by endomorphisms of  $AT$ -algebras of real rank zero contain classifiable Kirchberg algebras with arbitrary  $K$ -theory. In particular, by Kirchberg-Phillips classification, every Kirchberg algebra is isomorphic to such a crossed product. Significantly, Elliott's classification of (not necessarily simple)  $AT$ -algebras of real rank zero [11] implies that all unital simple  $AT$ -algebras of real rank zero with  $K_1$  equal to integers are modeled by crossed products associated to Cantor systems, studied by Putnam [48], see also [17]. Another milestone in the classification of non-simple  $C^*$ -algebras is Kirchberg's classification of strongly purely infinite, nuclear, separable  $C^*$ -algebras via ideal related  $KK$ -theory [26]. Nevertheless, this invariant is fairly complicated and there is still a lot of effort put into classifying certain non-simple purely infinite  $C^*$ -algebras by means of apparently less elaborated invariants, cf. [41], [7], [50]. Accordingly, it is of interest to establish non-trivial but still accessible examples of  $C^*$ -algebras whose ideal structure and  $K$ -theory of all ideals

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and quotients can be controlled. An overall aim of the present paper is to develop tools to construct and analyze a large class of crossed products by endomorphisms that fulfill these requirements. Another source of motivation comes from potential applications to spectral analysis of certain non-local operators [4], [22], [5].

In section 3, we introduce and study  $C_0(X)$ -dynamical systems. A  $C_0(X)$ -*dynamical system* is a pair  $(A, \alpha)$  where  $A$  is a  $C_0(X)$ -algebra and  $\alpha : A \rightarrow A$  is an endomorphism compatible with the  $C_0(X)$ -structure (we give several characterizations of this notion). Such a system can be viewed as a convenient combination of topological and noncommutative dynamics; encoded in a pair  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  where  $\varphi : \Delta \rightarrow X$  is a continuous proper mapping defined on an open set  $\Delta \subseteq X$ , and  $\alpha_x : A(\varphi(x)) \rightarrow A(x)$ ,  $x \in \Delta$ , is a homomorphism between the corresponding fibers of the  $C_0(X)$ -algebra  $A$ , so that

$$\alpha(a)(x) = \alpha_x(a(\varphi(x))), \quad a \in A, x \in \Delta.$$

We refer to the pair  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  as to a *morphism* of the corresponding  $C^*$ -bundle  $\mathcal{A} := \bigsqcup_{x \in X} A(x)$  (Definition 3.1). In particular, if every fiber  $A(x)$  is trivial (equal to  $\mathbb{C}$ ) we get a topological dynamical system. In the case when  $X$  is trivial (a singleton),  $\alpha : A \rightarrow A$  is just an endomorphism of  $A$ , and we call  $(A, \alpha)$  simply a  $C^*$ -*dynamical system*. An important non-trivial example arises when  $A$  is a unital  $C^*$ -algebra,  $C \subseteq Z(A)$  is a non-degenerate  $C^*$ -subalgebra of the center of  $A$  and  $\alpha$  ‘almost preserves’  $C$ , that is  $\alpha(C) \subseteq C\alpha(1)$ . Then  $(A, \alpha)$  is naturally a  $C_0(X)$ -dynamical system with  $C_0(X) \cong C$ . Analysis of crossed products associated to such  $C_0(X)$ -dynamical systems, in the case  $\alpha$  is an automorphism, played an important role in the study of non-local operators, cf. [22], such as (abstract) weighted shift operators [4], or singular integral operators with shifts [5]. If  $C = Z(A)$  and the primitive ideal space  $\text{Prim}(A)$  of  $A$  is a Hausdorff space, then  $X \cong \text{Prim}(A)$ , and this will be our model example.

We associate to any  $C^*$ -dynamical system  $(A, \alpha)$  and an ideal  $J$  in  $A$  the *relative crossed product*  $C^*(A, \alpha; J)$  introduced (for an arbitrary completely positive map) in [35]. As explained in detail in [35, Section 3.4], these crossed products include as special cases those studied in [44], [54], [40], [37], [33]. In the present paper, we consider only the case when  $A$  embeds into  $C^*(A, \alpha; J)$ , equivalently when  $J$  is contained in the annihilator  $(\ker \alpha)^\perp$  of the kernel of  $\alpha$  (the general case may be covered by passing to a quotient  $C^*$ -dynamical system, cf. Remark 2.8 below). The *unrelative crossed product* is  $C^*(A, \alpha) := C^*(A, \alpha; (\ker \alpha)^\perp)$ . If  $(A, \alpha)$  is a  $C_0(X)$ -dynamical system with the related morphism  $(\varphi, \{\alpha_x\}_{x \in \Delta})$ , then among our main results we list the following:

- *Isomorphism theorem.* We show that for certain continuous  $C_0(X)$ -algebras, if the map  $\varphi$  is topologically free outside a set  $Y$  related to the ideal  $J$ , then every injective representation of  $(A, \alpha)$  whose ideal of covariance is maximal possible, give rise to a faithful representation of  $C^*(A, \alpha; J)$  (see Theorem 4.11).

- *Description of the ideal structure.* We prove that, if  $\varphi$  is free, then we have a bijective correspondence between ideals  $\mathcal{I}$  in  $C^*(A, \alpha; J)$  and certain pairs  $(I, I')$  of ideals in  $A$ , called *J-pairs* for  $(A, \alpha)$  (see Theorem 4.12 and Definition 2.17). Moreover, the quotient of  $C^*(A, \alpha; J)$  by  $\mathcal{I}$  is naturally isomorphic a crossed product associated to the quotient of  $(A, \alpha)$ , and the ideal  $\mathcal{I}$  is Morita-Rieffel (strongly Morita) equivalent either to the crossed product associated to the restricted endomorphism  $\alpha|_I$  or to an endomorphism constructed

from  $(I, I')$  and  $\alpha$  (see Theorem 2.19 and Proposition 2.25). In the case  $A$  has a Hausdorff primitive ideal space and  $X = \text{Prim}(A)$ , we describe ideals in  $C^*(A, \alpha; J)$  in terms of pairs  $(V, V')$  of closed subsets of  $X$ , called *Y-pairs* for  $(X, \varphi)$  (see Proposition 5.6 and Definition 5.1). Hence the ideal structure of  $C^*(A, \alpha; J)$  is completely described in terms of the topological dynamical system  $(X, \varphi)$ . In particular, in this case we characterize simplicity of crossed products (Proposition 5.9).

• *Pure infiniteness.* It seems that amongst the existing technics of showing pure infiniteness of crossed products there are two types of approaches. In the first one the corresponding crossed product is simple [39], [19], [20]. In the second one the initial algebra  $A$  is assumed to have the ideal property [52], [16], [43], [46], [38]. We cover these two lines of research in our context by showing that if  $\varphi$  is free, then

$$\begin{array}{l} A \text{ is purely infinite and} \\ \text{has the ideal property} \end{array} \implies \text{the same is true for } C^*(A, \alpha, J).$$

and

$$\begin{array}{l} A \text{ is purely infinite and there are} \\ \text{finitely many } J\text{-pairs for } (A, \alpha) \end{array} \implies \begin{array}{l} C^*(A, \alpha, J) \text{ is purely infinite and} \\ \text{has finitely many ideals} \end{array}$$

(see Theorem 4.12). If  $X = \text{Prim}(A)$  and  $A$  is purely infinite, this leads us to necessary and sufficient conditions for  $C^*(A, \alpha)$  to be a Kirchberg algebra (Corollary 5.10). We recall that in the presence of the ideal property, pure infiniteness is equivalent to strong pure infiniteness. We also point out that using conditions introduced recently in [38], the aforementioned results could be potentially generalized to the case when  $A$  is not necessarily purely infinite. Moreover, in recent papers [28], [29] Sierakowski and Kirchberg introduced a new machinery that gives strong pure infiniteness criteria for crossed products by discrete group actions, without passing (explicitly) through pure infiniteness. It seems plausible that combining their technics with tools of the present paper one could also obtain permanence results for strong pure infiniteness of  $C^*(A, \alpha, J)$ . Nevertheless, we do not pursue these issues here.

• *K-theory.* In the case when the corresponding  $C^*$ -bundle is trivial, that is when  $A = C_0(X, D)$  for a  $C^*$ -algebra  $D$ , and under the assumptions that  $X$  is totally disconnected,  $K_0(D)$  is torsion free and  $K_1(D) = 0$ , we give formulas for  $K$ -groups of  $C^*(A, \alpha, J)$  formulated in terms of  $(X, \varphi, \{\alpha_x\}_{x \in \Delta}; Y)$  (Proposition 5.13). These formulas can be viewed as a far reaching generalization of those given by Putnam in [48]. If additionally  $D$  is simple and  $\varphi$  is free we get formulas for  $K$ -groups of all ideals in  $C^*(A, \alpha, J)$  (Theorem 5.15). We show by concrete examples that not only the dynamical system  $(X, \varphi)$  (which determines the ideal structure of  $C^*(A, \alpha, J)$ ) but also endomorphisms  $\alpha_x$ ,  $x \in \Delta$ , contribute to  $K$ -theory, thus giving us a lot of flexibility in constructing interesting algebras.

The aforementioned results are based on general facts for crossed products  $C^*(A, \alpha; J)$ , which we develop in section 2. One of the main tools is a description of a reversible  $J$ -extension  $(B, \beta)$  of  $(A, \alpha)$ , introduced in [33]: we show that if  $(A, \alpha)$  is a  $C_0(X)$ -dynamical system, then  $(B, \beta)$  is a  $C_0(\tilde{X})$ -dynamical system induced by a morphism  $(\tilde{\varphi}, \{\beta_x\}_{x \in \tilde{\Delta}})$  where  $(X, \tilde{\varphi})$  is a reversible  $Y$ -extension of  $(X, \varphi)$  introduced in [30] (see Theorem 4.9).

It has to be emphasized that so far, cf. [44], [54], [40], [37], [33], crossed products by endomorphisms were studied either in the case  $A$  is unital or under the assumption that

the endomorphism  $\alpha : A \rightarrow A$  is *extendible* [1], i.e. that it extends to an endomorphism of the multiplier algebra  $M(A)$  of  $A$ . However, these assumptions exclude a number of important applications. For instance, a restriction of an extendible endomorphism to an invariant ideal in general is not extendible. Thus, we are forced to develop a large part of theory of crossed products by *not necessarily extendible endomorphisms*. We do it in section 2. The established results are interesting in their own right.

More specifically, we generalize one of the main results of [33] and describe the gauge-invariant ideals  $\mathcal{I}$  in  $C^*(A, \alpha; J)$  by  $J$ -pairs  $(I, I')$  of ideals in  $A$ . Additionally, we show that  $\mathcal{I}$  is Morita-Rieffel equivalent either to the crossed product associated to the restricted endomorphism  $\alpha|_I$  or to an endomorphism constructed from  $(I, I')$  and  $\alpha$  (Theorem 2.19 and Proposition 2.25). We generalize the classic Pimsner-Voiculescu sequence so that it applies to the crossed product  $C^*(A, \alpha; J)$  (Proposition 2.26). As a consequence we get six-term exact sequences for  $K$ -groups of all gauge-invariant ideals in  $C^*(A, \alpha; J)$  (Theorem 2.27). We extend the terminology of [33] and say that  $(A, \alpha)$  is a *reversible  $C^*$ -dynamical system* if  $\alpha$  has a complemented kernel and a hereditary range. For an arbitrary  $C^*$ -dynamical system  $(A, \alpha)$  we generalize the construction of a *reversible  $J$ -extension*  $(B, \beta)$  introduced in [33], see also [36]. We show that

$$C^*(A, \alpha; J) \cong C^*(B, \beta)$$

(Theorem 2.29). This is a powerful tool because for reversible systems  $(A, \alpha)$  the crossed product  $C^*(A, \alpha)$  has an accessible structure, very similar to that of classical crossed product by an automorphism. In particular, for such systems we have natural criteria for uniqueness property, gauge-invariance of all ideals, and pure infiniteness of  $C^*(A, \alpha)$  (see Propositions 2.35 and 2.46).

We note that, in contrast to [33] where more direct methods were used, in the present paper we base our more general analysis on certain results for relative Cuntz-Pimsner algebras and an identification of  $C^*(A, \alpha; J)$  as such an algebra. We present the relevant facts in Appendix A.

The content is organized as follows: We recall the relevant notions and facts concerning  $C_0(X)$ -algebras in section 1. General crossed products are studied in section 2. In section 3 we introduce and analyze  $C_0(X)$ -dynamical systems. Section 4 contains general main results for  $C_0(X)$ -dynamical systems. We apply them to  $C_0(X)$ -dynamical systems with  $X = \text{Prim}(A)$  in section 5, where our results attain a particularly nice form. We finish the paper with Appendix A, which contains relevant facts from the theory of  $C^*$ -correspondences and relative Cuntz-Pimsner algebras, as well as a discussion of a particular case of the  $C^*$ -correspondence  $E_\alpha$  associated to  $(A, \alpha)$ .

**0.1. Notation and conventions.** The set of natural numbers  $\mathbb{N}$  starts from zero. All ideals in  $C^*$ -algebras are assumed to be closed and two-sided. All homomorphisms between  $C^*$ -algebras are by definition  $*$ -preserving. For actions  $\gamma : A \times B \rightarrow C$  such as multiplications, inner products, etc., we use the notation:

$$\gamma(A, B) = \overline{\text{span}}\{\gamma(a, b) : a \in A, b \in B\}.$$

If  $A$  is a  $C^*$ -algebra  $1$  denotes the unit in the multiplier  $C^*$ -algebra  $M(A)$ . The enveloping von Neumann algebra of  $A$  is denoted by  $A^{**}$ . We recall, see [27, Theorem 4.16], that

a  $C^*$ -algebra  $A$  is *purely infinite* if and only if every  $a \in A^+ \setminus \{0\}$  is *properly infinite*, e.g.  $a \oplus a \precsim a \oplus 0$  in  $M_2(A)$ , where  $\precsim$  Cuntz comparison of positive elements. We recall that for  $a, b \in A^+$ ,  $a \precsim b$  in  $A$  if and only if for every  $\varepsilon > 0$  there is  $x \in A^+$  such that  $\|a - xbx\| < \varepsilon$ . A  $C^*$ -algebra  $A$  has *the ideal property* [47], [45], if every ideal in  $A$  is generated (as an ideal) by its projections.

## 1. PRELIMINARIES ON $C_0(X)$ -ALGEBRAS AND $C^*$ -BUNDLES

In this section, we gather certain facts concerning  $C_0(X)$ -algebras. We find it beneficial to use two equivalent pictures of such objects: as  $C^*$ -algebras with a  $C_0(X)$ -module structure and as  $C^*$ -algebras of sections of  $C^*$ -bundles. Thus we implement both of these viewpoints. As a general reference we use [55, Section C], but cf. also, for instance, [18], [6].

**1.1.  $C^*$ -bundles and section  $C^*$ -algebras.** Let  $X$  be a locally compact Hausdorff space. An *upper semicontinuous  $C^*$ -bundle over  $X$*  is a topological space  $\mathcal{A} = \bigsqcup_{x \in X} A(x)$  such that the natural surjection  $p : \mathcal{A} \rightarrow X$  is open continuous, each fiber  $A(x)$  is a  $C^*$ -algebra, the mapping  $\mathcal{A} \ni a \rightarrow \|a\| \in \mathbb{R}$  is upper semicontinuous, and the  $*$ -algebraic operations in each of the fibers are continuous in  $\mathcal{A}$ , for details see [55, Definition C.16]. If additionally, the mapping  $\mathcal{A} \ni a \rightarrow \|a\| \in \mathbb{R}$  is continuous,  $\mathcal{A}$  is called a *continuous  $C^*$ -bundle over  $X$* . For each  $x \in X$ , we denote by  $0_x$  the zero element in the fiber  $C^*$ -algebra  $A(x)$ , and by  $1_x$  the unit in the multiplier algebra  $M(A(x))$  of  $A(x)$ . A  $C^*$ -bundle  $\mathcal{A}$  is *trivial* if there is a  $C^*$ -algebra  $D$  and homeomorphism from  $\mathcal{A} = \bigsqcup_{x \in X} A(x)$  onto  $X \times D$  which intertwines  $p$  with the projection onto the first coordinate.

We denote by  $\Gamma(\mathcal{A}) := \{a \in C(X, \mathcal{A}) : p(a(x)) = x\}$  the set of continuous sections of the upper semicontinuous  $C^*$ -bundle  $\mathcal{A}$ . It is a  $*$ -algebra with respect to natural pointwise operations. Moreover, the set of continuous sections that vanish at infinity

$$\Gamma_0(\mathcal{A}) := \{a \in \Gamma(\mathcal{A}) : \forall_{\varepsilon > 0} \quad \{x \in X : \|a(x)\| \geq \varepsilon\} \text{ is compact}\}$$

is a  $C^*$ -algebra with the norm  $\|a\| := \sup_{x \in X} \|a(x)\|$ . We call  $\Gamma_0(\mathcal{A})$  the *section  $C^*$ -algebra of  $\mathcal{A}$* . The section algebra  $\Gamma_0(\mathcal{A})$  determines the topology of the  $C^*$ -bundle  $\mathcal{A}$ . In particular, we have the following lemma (see, for instance, the proof of [55, Theorem C.25]).

**Lemma 1.1.** *A net  $\{b_i\}$  converges to  $b$  in the  $C^*$ -bundle  $\mathcal{A}$  if and only if  $p(b_i) \rightarrow p(b)$  and for each  $\varepsilon > 0$  there is  $a \in \Gamma_0(\mathcal{A})$  such that  $\|a(p(b)) - b\| < \varepsilon$  and we eventually have  $\|a(p(b_i)) - b_i\| < \varepsilon$ .*

The algebra  $\Gamma_0(\mathcal{A})$  is naturally equipped with the structure of  $C_0(X)$ -algebra given by  $(f \cdot a)(x) := f(x)a(x)$  for  $f \in C_0(X)$  and  $a \in \Gamma_0(\mathcal{A})$ .

**1.2.  $C_0(X)$ -algebras.** A  $C_0(X)$ -algebra is a  $C^*$ -algebra  $A$  endowed with a nondegenerate homomorphism  $\mu_A$  from  $C_0(X)$  into the center  $Z(M(A))$  of the multiplier algebra  $M(A)$  of  $A$ . When  $X$  is compact  $A$  is also called a  $C(X)$ -algebra. The  $C_0(X)$ -algebra  $A$  is viewed as a  $C_0(X)$ -module where

$$f \cdot a := \mu_A(f)a, \quad f \in C_0(X), \quad a \in A.$$

Accordingly, the *structure map*  $\mu_A : C_0(X) \rightarrow Z(M(A))$  is often suppressed. Using the Dauns-Hofmann isomorphism we may identify  $Z(M(A))$  with  $C_b(\text{Prim } A)$ , and then  $\mu_A$

becomes the operator of composition with a continuous map  $\sigma_A : \text{Prim } A \rightarrow X$ . This map, called the *base map*, is determined by the equivalence:

$$(1) \quad C_0(X \setminus \{x\}) \cdot A \subseteq P \iff \sigma_A(P) = x, \quad P \in \text{Prim}(A).$$

Let us fix a  $C_0(X)$ -algebra  $A$  and consider a bundle  $\mathcal{A} := \bigsqcup_{x \in X} A(x)$  where

$$A(x) := A / (C_0(X \setminus \{x\}) \cdot A), \quad x \in X.$$

It can be shown that there is a unique topology on  $\mathcal{A} := \bigsqcup_{x \in X} A(x)$  such that  $\mathcal{A}$  becomes an upper semicontinuous  $C^*$ -bundle and the  $C_0(X)$ -algebra  $A$  can be identified with  $\Gamma_0(\mathcal{A})$  by writing  $a(x)$  for the image of  $a \in A$  in the quotient algebra  $A(x)$ . Moreover,  $\mathcal{A}$  is a continuous  $C^*$ -bundle if and only if  $\sigma_A : \text{Prim } A \rightarrow X$  is an open map. In the latter case,  $A$  is called a *continuous  $C_0(X)$ -algebra*. In other words, we have the following statement, see [55, Theorem C.26].

**Theorem 1.2.** *A  $C^*$ -algebra  $A$  is a  $C_0(X)$ -algebra if and only if  $A \cong \Gamma_0(\mathcal{A})$  where  $\mathcal{A}$  is an upper semicontinuous  $C^*$ -bundle. Moreover,  $A$  is a continuous  $C_0(X)$ -algebra if and only if  $\mathcal{A}$  is a continuous  $C^*$ -bundle.*

**Convention 1.3.** In the sequel we will freely pass (often without a warning) between the above equivalent descriptions. Thus for any  $C_0(X)$ -algebra  $A$  we will write  $A = \Gamma_0(\mathcal{A})$  where  $\mathcal{A}$  is the associated  $C^*$ -bundle.

**Remark 1.4.** Let  $A$  be a  $C_0(X)$ -algebra. In view of (1), we have  $\bigcap_{P \in \sigma_A^{-1}(x)} P = C_0(X \setminus \{x\}) \cdot A$  whenever  $x \in \sigma_A(\text{Prim } A)$ , and  $A(x) = \{0\}$  if and only if  $x \notin \sigma_A(\text{Prim } A)$ . Thus if  $\sigma_A(\text{Prim}(A))$  is locally compact (which is always the case when  $A$  is unital, or when  $A$  is a continuous  $C_0(X)$ -algebra), then we may treat  $A$  as a  $C_0(\sigma_A(\text{Prim}(A)))$ -algebra; in other words, we may assume that  $\sigma_A$  is surjective, or equivalently that all fibers  $A(x)$  are non-trivial.

**1.3. Multiplier algebra of a  $C_0(X)$ -algebra.** We say that a  $C_0(X)$ -algebra  $A$  has *local units* if all fibers  $A(x)$ ,  $x \in X$ , are unital, and for any  $x \in X$  there is  $a \in A$  such that  $a(y) = 1_y$  is the unit in  $A(y)$  for all  $y$  in a neighborhood of  $x$ .

**Lemma 1.5.** *A  $C_0(X)$ -algebra  $A$  is unital if and only if  $A$  has local units and the range of  $\sigma_A$  is compact.*

*Proof.* If 1 is the unit in  $A$  then  $\sigma_A(\text{Prim}(A)) = \{x \in X : \|1(x)\| \geq 1/2\}$  is compact, because  $1 \in \Gamma_0(\mathcal{A})$ , and clearly the (global) unit 1 is a local unit for any point in  $X$ . Conversely, suppose that  $\sigma_A(\text{Prim}(A))$  is compact and  $A$  has local units. Consider the function  $1 : X \rightarrow \mathcal{A} = \bigsqcup_{x \in X} A(x)$  where for each  $x \in A$  we let  $1(x) := 1_x$  to be the unit in  $A(x)$ . Using Lemma 1.1 and local units one readily sees that 1 is a continuous section of  $\mathcal{A}$ . For any  $\varepsilon$  the set  $\{x \in X : \|1(x)\| \geq \varepsilon\}$  is compact, as a closed subset of  $\sigma_A(\text{Prim}(A))$ . Thus  $1 \in \Gamma_0(\mathcal{A}) = A$ .  $\square$

We have the following natural description of the multiplier algebra  $M(A)$  of a  $C_0(X)$ -algebra  $A$  as sections of the set  $M(\mathcal{A}) := \bigsqcup_{x \in X} M(A(x))$ , see [55, Lemma C.11]. We emphasize

however, that in general (even when  $X$  is compact)  $M(\mathcal{A})$  can not be equipped with a topology making it an upper semicontinuous  $C^*$ -bundle such that  $M(A) \subseteq \Gamma(M(\mathcal{A}))$ , see [55, Example C.13].

**Proposition 1.6.** *Suppose that  $A$  is a  $C_0(X)$ -algebra. The multiplier algebra  $M(A)$  can be naturally identified with the set of all functions  $m$  on  $X$  such that  $m(x) \in M(A(x))$ , for all  $x \in X$ , and the functions  $x \mapsto m(x)a(x)$ ,  $x \mapsto a(x)m(x)$  are in  $A = \Gamma_0(\mathcal{A})$  for any  $a \in A$ . Then the  $C^*$ -algebraic structure of  $M(A)$  is given by the pointwise operations and the supremum norm  $\|m\| = \sup_{x \in X} \|m(x)\|$ .*

**1.4. Ideals and quotients of a  $C_0(X)$ -algebra.** Fix a  $C_0(X)$ -algebra  $A$  and let  $I$  be an ideal in  $A$ . Assuming the standard identifications  $\text{Prim } I = \{P \in \text{Prim } A : I \not\subseteq P\}$  and  $\text{Prim}(A/I) = \{P \in \text{Prim } A : I \subseteq P\}$ , we see that both  $I$  and  $A/I$  are  $C_0(X)$ -algebras with base maps  $\sigma_A : \text{Prim}(I) \rightarrow X$  and  $\sigma_A : \text{Prim}(A/I) \rightarrow X$  respectively. Moreover, we have natural isomorphisms  $(A/I)(x) \cong A(x)/I(x)$  where  $I(x) = \{a(x) : a \in I \subseteq A\}$ ,  $x \in X$ .

Suppose that  $A$  is a continuous  $C_0(X)$ -algebra. Then the ideal  $I$  is naturally a continuous  $C_0(Y)$ -algebra for any locally compact set  $Y$  containing the open set  $\sigma_A(\text{Prim}(I))$ , because a restriction of an open map to an open set is open (independently of the codomain). The situation is quite different when dealing with a restriction to a closed set, and thus the case of the quotient  $A/I$  is more delicate. Nevertheless, the set  $Y = \sigma_A(\text{Prim}(A/I))$  is locally compact, and the mapping  $\sigma_A : \text{Prim}(A/I) \rightarrow Y$  is open for instance when  $I$  is complemented or  $\sigma_A$  is injective. Translating this to the language of  $C^*$ -bundles we get the following lemma.

**Lemma 1.7.** *Suppose that  $I$  is an ideal in a  $C^*$ -algebra  $A = \Gamma_0(\mathcal{A})$  of continuous sections of an upper semicontinuous  $C^*$ -bundle  $\mathcal{A} = \bigsqcup_{x \in X} A(x)$ . The ideal  $I$  and the quotient algebra  $A/I$  can be naturally treated as algebras of continuous sections of  $\mathcal{I} = \bigsqcup_{x \in X} I(x)$  and  $\mathcal{A}/\mathcal{I} = \bigsqcup_{x \in X} A(x)/I(x)$  (equipped with unique topologies), respectively. Moreover, we have*

$$(2) \quad \{x \in X : I(x) \neq \{0\}\} = \sigma_A(\text{Prim}(I)),$$

$$(3) \quad \{x \in X : I(x) \neq A(x)\} = \sigma_A(\text{Prim}(A/I)).$$

*If  $\mathcal{A}$  is a continuous bundle, then  $\mathcal{I}$  is continuous over the set (2) and  $\mathcal{A}/\mathcal{I}$  is continuous over the set (3) whenever  $I$  is complemented or  $\sigma_A$  is injective.*

*Proof.* In view of the above discussion we only need to show (2) and (3). The equivalences

$$I(x) \neq \{0\} \iff I \not\subseteq \bigcap_{P \in \sigma_A^{-1}(x)} P \iff \exists_{P \in \sigma_A^{-1}(x)} I \not\subseteq P \iff x \in \sigma_A(\text{Prim}(I))$$

prove (2). To see (3) notice that using (1) we get

$$\begin{aligned} I(x) \neq A(x) &\iff \left( I + \bigcap_{P \in \sigma_A^{-1}(x)} P \right) \neq A \iff \exists_{P_0 \in \text{Prim}(A)} \left( I + \bigcap_{P \in \sigma_A^{-1}(x)} P \right) \subseteq P_0 \\ &\iff \exists_{P_0 \in \text{Prim}(A)} I \subseteq P_0 \text{ and } \bigcap_{P \in \sigma_A^{-1}(x)} P \subseteq P_0 \\ &\iff x \in \sigma_A(\text{Prim}(A/I)). \end{aligned}$$

□

Let  $I$  be an ideal in the  $C_0(X)$ -algebra  $A$ . The annihilator  $I^\perp = \{a \in A : aI = 0\}$  of  $I$  is also a  $C_0(X)$ -algebra with the base map  $\sigma_A : \text{Prim}(I^\perp) \rightarrow X$ . Moreover, since  $I^\perp$  is the biggest ideal in  $A$  with the property that  $I \cap I^\perp = \{0\}$  it follows that  $\text{Prim}(I^\perp) = \text{Int}(\text{Prim}(A/I))$ . If  $A$  is a continuous  $C_0(X)$ -algebra then  $I^\perp$  is a continuous  $C_0(U)$ -algebra where  $U = \sigma_A(\text{Int}(\text{Prim}(A/I)))$ . In terms of  $C^*$ -bundles,  $I^\perp$  can be viewed as the algebra of continuous sections of the  $C^*$ -bundle  $\mathcal{I}^\perp := \bigsqcup_{x \in X} I(x)^\perp$  where  $I(x)^\perp$  is contained in the annihilator of  $I(x)$  in  $A(x)$ . In particular,  $I(x)^\perp = \{0\}$  if and only if  $x \notin U$ .

## 2. GENERAL CROSSED PRODUCTS BY ENDOMORPHISMS

In this section, we define crossed products  $C^*(A, \alpha; J)$  for an arbitrary (not necessarily extendible) endomorphism  $\alpha : A \rightarrow A$ . We establish basic results concerning the structure of these  $C^*$ -algebras, including description of all gauge-invariant ideals and ‘Pimsner Voiculescu sequences’ determining their  $K$ -theory. We also construct a reversible  $J$ -extension  $(B, \beta)$  of  $(A, \alpha)$ , discuss the notion of topological freeness for systems which are reversible or commutative, and give a general pure infiniteness criteria for reversible systems.

**2.1.  $C^*$ -dynamical systems and their crossed products.** A  $C^*$ -dynamical system is a pair  $(A, \alpha)$  where  $A$  is a  $C^*$ -algebra and  $\alpha : A \rightarrow A$  is an endomorphism. We say that  $\alpha$ , or that the system  $(A, \alpha)$ , is *extendible* [1] if  $\alpha$  extends to a strictly continuous endomorphism  $\bar{\alpha} : M(A) \rightarrow M(A)$ . It is known to hold exactly when for some (and hence any) approximate unit  $\{\mu_\lambda\}$  in  $A$  the net  $\{\alpha(\mu_\lambda)\}$  converges strictly in  $M(A)$ . In contrast to [33], in the present paper in general we do not assume that  $(A, \alpha)$  is extendible.

**Definition 2.1** (Definition 2.4 in [33]). A  $C^*$ -dynamical system  $(A, \alpha)$  is called *reversible* if  $\ker \alpha$  is a complemented ideal in  $A$  and  $\alpha(A)$  is a hereditary subalgebra of  $A$  (briefly,  $\alpha$  has a complemented kernel and a hereditary range).

**Remark 2.2.** An extendible endomorphism  $\alpha : A \rightarrow A$  has a hereditary range if and only if it is a *corner endomorphism*, that is if  $\alpha(A)$  is a corner in  $A$  (we then necessarily have  $\bar{\alpha}(1)A\bar{\alpha}(1) = \alpha(A)$ ). In particular, an extendible  $C^*$ -dynamical system  $(A, \alpha)$  is reversible if and only if  $\alpha$  is a corner endomorphism with complemented kernel.

Suppose that  $(A, \alpha)$  is a reversible  $C^*$ -dynamical system. Then  $\alpha : (\ker \alpha)^\perp \rightarrow \alpha(A)A\alpha(A)$  is an isomorphism and we denote its inverse by  $\alpha^{-1}$ . If  $(A, \alpha)$  is extendible, then  $\alpha(A)A\alpha(A) = \bar{\alpha}(1)A\bar{\alpha}(1)$  and  $\alpha^{-1}$  extends to a completely positive map  $\alpha_* : A \rightarrow A$  given by the formula

$$(4) \quad \alpha_*(a) = \alpha^{-1}(\bar{\alpha}(1)a\bar{\alpha}(1)), \quad a \in A.$$

The map  $\alpha_*$  is a *transfer operator* for  $(A, \alpha)$  in the sense of Exel [13], that is we have  $\alpha_*(\alpha(a)b) = a\alpha_*(b)$ , for all  $a, b \in A$ . Moreover,  $\alpha_*$  is *regular*, which means that  $\alpha \circ \alpha_*$  is a conditional expectation onto  $\alpha(A)$ . In fact,  $\alpha_*$  is a unique *regular transfer operator* for  $(A, \alpha)$ , see [35, Proposition 4.15]. Transfer operators satisfying (4) appear in a natural way in a number of papers, see for instance [13], [2], [34], [33].



**Example 2.3.** If  $A = C_0(X)$  where  $X$  is a locally compact Hausdorff space, then every endomorphism  $\alpha : A \rightarrow A$  is of the form

$$\alpha(a)(x) = \begin{cases} a(\varphi(x)), & x \in \Delta, \\ 0, & x \notin \Delta, \end{cases}$$

where  $\varphi : \Delta \rightarrow X$  is a continuous proper mapping defined on an open subset  $\Delta \subseteq X$ . Note that, properness of  $\varphi$  implies that  $\varphi(\Delta)$  is closed in  $X$ . We call the pair  $(X, \varphi)$  a *partial dynamical system* dual to  $(A, \alpha)$ . The endomorphism  $\alpha$  is extendible if and only if  $\Delta$  is closed. The kernel of  $\alpha$  is a complemented ideal in  $A$  if and only if  $\varphi(\Delta)$  is open. The pair  $(A, \alpha)$  is a reversible  $C^*$ -dynamical system if and only if both  $\Delta$  and  $\varphi(\Delta)$  are open in  $X$  and  $\varphi : \Delta \rightarrow \varphi(\Delta)$  is a homeomorphism. If the latter conditions are satisfied we say that  $(X, \varphi)$  is a *reversible partial dynamical system*.

Now, we turn to the definition of crossed products. For more details, in the case  $A$  is unital or  $\alpha$  is extendible, see [37] and [33].

**Definition 2.4.** A *representation*  $(\pi, U)$  of a  $C^*$ -dynamical system  $(A, \alpha)$  on a Hilbert space  $H$  consists of a non-degenerate representation  $\pi : A \rightarrow \mathcal{B}(H)$  and an operator  $U \in \mathcal{B}(H)$  such that

$$(5) \quad U\pi(a)U^* = \pi(\alpha(a)), \quad \text{for all } a \in A.$$

We will occasionally deal with representations of  $(A, \alpha)$  in a  $C^*$ -algebra  $B$  by which mean a pair  $(\pi, U)$  where  $\pi : A \rightarrow B$  is a non-degenerate homomorphism and  $U \in B^{**}$  (an element of the enveloping von Neumann algebra of  $B$ ) satisfies (5). If  $\pi$  is injective then we say  $(\pi, U)$  is *injective*.

Let  $(\pi, U)$  be a representation of  $(A, \alpha)$  in a  $C^*$ -algebra  $B$ . Then  $U$  is necessarily a partial isometry. Indeed, if  $\{\mu_\lambda\}$  is an approximate unit in  $A$  then by non-degeneracy of  $\pi$ ,  $\{\pi(\mu_\lambda)\}$  converges  $\sigma$ -weakly to the unit in  $B^{**}$  and therefore  $\{\pi(\alpha(\mu_\lambda))\} = \{U\pi(\mu_\lambda)U^*\}$  converges  $\sigma$ -weakly to  $UU^*$ . Hence, using multiplicativity of  $\alpha$ , we get that

$$(UU^*)^2 = \sigma\text{-}\lim_{\lambda} \sigma\text{-}\lim_{\lambda'} \pi(\alpha(\mu_\lambda))\pi(\alpha(\mu_{\lambda'})) = \sigma\text{-}\lim_{\lambda} \pi(\alpha(\mu_\lambda)) = U^*U$$

is a projection, cf. [35, Proposition 3.21]. Moreover, see [35, Proposition 3.21] or the proof of [37, Lemma 1.2], multiplicativity of  $\alpha$  implies that the initial projection  $U^*U$  of  $U$  commutes with the elements of  $\pi(A)$ . In particular,

$$I_{(\pi, U)} := \{a \in A : U^*U\pi(a) = \pi(a)\}$$

is an ideal in  $A$ . If an ideal  $J$  in  $A$  is contained in  $I_{(\pi, U)}$  we say that the representation  $(\pi, U)$  is  *$J$ -covariant*. If  $(\pi, U)$  is  $(\ker \alpha)^\perp$ -covariant, that is if

$$a \in (\ker \alpha)^\perp \implies \pi(a) = U^*U\pi(a)$$

we say that  $(\pi, U)$  is a *covariant representation*. Note that if  $\alpha$  is injective, then the representation  $(\pi, U)$  is covariant if and only if  $U$  is an isometry. Thus if  $\alpha$  is injective and non-degenerate then the representation  $(\pi, U)$  is covariant if and only if  $U$  is a unitary. The special role of  $(\ker \alpha)^\perp$  is also indicated in the following fact.

**Lemma 2.5.** Suppose that  $(\pi, U)$  is an injective representation of  $(A, \alpha)$ . Then  $I_{(\pi, U)} \subseteq (\ker \alpha)^\perp$ . In particular,  $I_{(\pi, U)} = (\ker \alpha)^\perp$  if and only if  $(\pi, U)$  is a covariant representation.

*Proof.* Let  $a \in I_{(\pi, U)}$  and  $b \in \ker \alpha$ . Then  $ab \in I_{(\pi, U)}$  and

$$\pi(ab) = U^*U\pi(ab)U^*U = U^*\pi(\alpha(ab))U = U^*\pi(\alpha(a))\pi(\alpha(b))U = 0.$$

Hence  $ab = 0$  because  $\pi$  is injective. Accordingly,  $I_{(\pi, U)} \subseteq (\ker \alpha)^\perp$ .  $\square$

Combining the above lemma and the following proposition one can see that  $(A, \alpha)$  admits an injective  $J$ -covariant representation if and only if  $J \subseteq (\ker \alpha)^\perp$ .

**Proposition 2.6.** *For any  $C^*$ -dynamical system  $(A, \alpha)$  and any ideal  $J$  in  $(\ker \alpha)^\perp$  there exists a  $C^*$ -algebra  $C^*(A, \alpha; J)$  containing  $A$  as a non-degenerate  $C^*$ -algebra and an operator  $u \in C^*(A, \alpha; J)^{**}$  such that*

a)  $C^*(A, \alpha; J)$  is generated (as a  $C^*$ -algebra) by  $A \cup uA$ ,

$$\alpha(a) = uau^* \text{ for each } a \in A \quad \text{and} \quad J = \{a \in A : u^*ua = a\},$$

b) for every  $J$ -covariant representation  $(\pi, U)$  of  $(A, \alpha)$  there is a representation  $\pi \rtimes U$  of  $C^*(A, \alpha; J)$  determined by relations  $(\pi \rtimes U)(a) = \pi(a)$ ,  $a \in A$ , and  $(\pi \rtimes U)(u) = U$ .

Moreover, if  $\alpha$  is extendible, then  $u \in M(C^*(A, \alpha; J))$  and  $C^*(A, \alpha; J) = C^*(A \cup Au)$ .

*Proof.* Existence of  $C^*(A, \alpha; J)$  with the prescribed properties can be deduced from Propositions A.8 and A.1. It also follows from [35, Proposition 3.26] which in essence states that  $C^*(A, \alpha; J)$  is a special case of the crossed product defined in [35, Definition 3.5] (note that the identification of the aforementioned algebras goes thorough the equality  $s = u^*$ ). In particular, [35, Remark 3.11] implies that  $u \in C^*(A, \alpha; J)^{**}$  and when  $\alpha$  is extendible then  $u \in M(C^*(A, \alpha; J))$ . If  $\alpha$  is extendible then  $C^*(A, \alpha; J) = C^*(A \cup Au)$  by [35, Lemma 3.23].  $\square$

Universal properties of the  $C^*$ -algebra  $C^*(A, \alpha; J)$  imply that, up to a natural isomorphism, it is uniquely determined by the triple  $(A, \alpha, J)$ .

**Definition 2.7.** We define the *relative crossed product* associated to a  $C^*$ -dynamical system  $(A, \alpha)$  and an ideal  $J$  in  $(\ker \alpha)^\perp$  to be the  $C^*$ -algebra described in Proposition 2.6. We also write  $C^*(A, \alpha) := C^*(A, \alpha, (\ker \alpha)^\perp)$  and call it the (unrelative) *crossed product of  $A$  by  $\alpha$* .

**Remark 2.8.** In the case  $A$  is unital or  $\alpha$  is extendible, the  $C^*$ -algebra  $C^*(A, \alpha; J)$  was studied respectively in [37] and [33]. In general, the crossed product  $C^*(A, \alpha; J)$  is a special case of the one defined in [35, Definition 3.5] where  $\alpha$  is treated as a completely positive map, or the one introduced in [31, Definition 4.9] where  $\alpha$  is treated as a partial morphism of  $A$ . In particular, one could consider the crossed product  $C^*(A, \alpha; J)$  for an arbitrary ideal  $J$  in  $A$ , not necessarily contained in  $(\ker \alpha)^\perp$ , cf. [35], or [31]. However, if  $J \not\subseteq (\ker \alpha)^\perp$  the algebra  $A$  does not embed into  $C^*(A, \alpha; J)$ . Moreover, as described in [37, Section 5.3], see also [31, Example 6.24], or [33, Remark 4.4], by passing to a quotient  $C^*$ -dynamical system, one can always reduce this seemingly more general situation to that of Definition 2.7.

By universal property of the crossed product  $C^*(A, \alpha; J)$ , there is a circle action  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \ni z \longmapsto \gamma_z \in \text{Aut}(C^*(A, \alpha; J))$  determined by relations  $\gamma_z(a) = a$ ,  $\gamma_z(u) = zu$ ,  $a \in A$ ,  $z \in \mathbb{T}$ . We call  $\gamma = \{\gamma_z\}_{z \in \mathbb{T}}$  the *gauge action* on  $C^*(A, \alpha; J)$ . We say that a representation  $(\pi, U)$  of  $(A, \alpha)$  admits a *gauge action* if the relations  $\gamma_z(\pi(a)) = \pi(a)$ ,

$\gamma_z(U) = zU$ ,  $a \in A$ ,  $z \in \mathbb{T}$ , determine a circle action on the  $C^*$ -algebra generated by  $\pi(A) \cup U\pi(A)$ . We have the following version of the gauge-uniqueness theorem.

**Proposition 2.9.** *For any injective  $J$ -covariant representation  $(\pi, U)$  the homomorphism  $\pi \rtimes U$  of  $C^*(A, \alpha; J)$  is injective if and only if  $I_{(\pi, U)} = J$  and  $(\pi, U)$  admits a gauge action.*

*In particular, if  $(\pi, U)$  is a covariant representation of  $(A, \alpha)$  then the homomorphism  $\pi \rtimes U$  of  $C^*(A, \alpha)$  is injective if and only if  $\pi$  is injective and  $(\pi, U)$  admits a gauge action.*

*Proof.* The first part of the assertion follows from Propositions A.2 and A.8. For the second part apply Lemma 2.5.  $\square$

We list certain general permanence properties for the crossed products  $C^*(A, \alpha; J)$ .

**Proposition 2.10.** *Let  $(A, \alpha)$  be a  $C^*$ -dynamical system and let ideal  $J$  be an ideal in  $(\ker \alpha)^\perp$ .*

- (i)  $A$  is exact  $\iff C^*(A, \alpha; J)$  is exact.
- (ii)  $A$  is nuclear  $\implies C^*(A, \alpha; J)$  is nuclear.
- (iii) If  $A$  is separable, nuclear, and both  $A$  and  $J$  satisfy the UCT, then  $C^*(A, \alpha; J)$  satisfies the UCT.

*Proof.* By Proposition A.8, we have  $C^*(A, \alpha; J) \cong \mathcal{O}(J, E_\alpha)$ . Since  $\mathcal{O}(J, E_\alpha)$  is the quotient of the Toeplitz algebra  $\mathcal{T}_{E_\alpha} = C(\{0\}, E_\alpha)$ , item (i) follows from [24, Theorem 7.1]. Similarly, [24, Theorem 7.2] implies (ii). The argument leading to [24, Proposition 8.8] gives (iii).  $\square$

**2.2. Algebraic structure of crossed products.** The  $*$ -algebraic structure underlying the crossed product  $C^*(A, \alpha; J)$ , that could actually be used to construct  $C^*(A, \alpha; J)$ , cf. [31, Example 2.20 and Definition 4.9], is described in the following proposition.

**Proposition 2.11.** *For any  $C^*$ -dynamical system  $(A, \alpha)$  and any ideal  $J$  in  $(\ker \alpha)^\perp$  the universal operator  $u \in C^*(A, \alpha; J)^{**}$  is a power partial isometry, that is  $\{u^{n*}u^n\}_{n \in \mathbb{N}}$  and  $\{u^n u^{n*}\}_{n \in \mathbb{N}}$  are decreasing sequences of mutually commuting projections. Moreover,*

$$(6) \quad u^n a = \alpha^n(a) u^n, \quad \text{for all } a \in A, n \in \mathbb{N},$$

*so the projections  $u^{*n}u^n$  commute with elements of  $A$ . The elements*

$$(7) \quad a = \sum_{n,m=1}^N u^{*n} a_{n,m} u^m, \quad a_{n,m} \in \alpha^n(A) A \alpha^m(A), \quad n, m = 1, \dots, N, \quad N \in \mathbb{N},$$

*form a dense  $*$ -subalgebra of  $C^*(A, \alpha; J)$  and their products are determined by the formula*

$$(8) \quad (u^{*n} a_{n,m} u^m) (u^{*m+k} a_{m+k,l} u^l) = u^{*n+k} \alpha^k(a_{n,m}) a_{m+k,l} u^l, \quad n, m, k, l \in \mathbb{N}.$$

*Proof.* The first part of the assertion follows from [35, Proposition 3.21] and the fact that if  $(\pi, U)$  is a representation of  $(A, \alpha)$ , then  $(\pi, U^n)$  is a representation of  $(A, \alpha^n)$ ,  $n \in \mathbb{N}$ . Let us show (8). Since  $a_{n,m} \in A \alpha^m(A)$  we have  $a_{n,m} u^m u^{*m} = a_{n,m}$  and by (6) we get  $a_{n,m} u^{*k} = u^{*k} \alpha^k(a_{n,m})$ . Thus

$$(u^{*n} a_{n,m} u^m) (u^{*m+k} a_{m+k,l} u^l) = u^{*n} a_{n,m} u^{*k} a_{m+k,l} u^l = u^{*n+k} \alpha^k(a_{n,m}) a_{m+k,l} u^l.$$

Now, using (8), one readily sees that elements (7) form a  $*$ -algebra generated by  $A$  and  $uA = \alpha(A)u$ .  $\square$

**Corollary 2.12.** *The initial projection  $u^*u$  of the universal partial isometry  $u \in C^*(A, \alpha; J)^{**}$  belongs to the multiplier algebra  $M(C^*(A, \alpha; J))$  of  $C^*(A, \alpha; J)$ .*

*Proof.* Let  $a_{n,m} \in \alpha^n(A)A\alpha^m(A)$ ,  $n, m \in \mathbb{N}$ . If  $n > 0$ , then  $(u^*u)u^{*n}a_{n,m}u^m = u^{*n}a_{n,m}u^m \in C^*(A, \alpha; J)$ . If  $n = 0$  then

$$(u^*u)u^{*n}a_{n,m}u^m = (u^*u)a_{0,m}u^m = u^*\alpha(a_{0,m})u^{m+1} \in C^*(A, \alpha; J).$$

By Proposition 2.11 we get  $(u^*u)C^*(A, \alpha; J) \subseteq C^*(A, \alpha; J)$  and consequently (since  $u^*u$  is self-adjoint)  $u^*u \in M(C^*(A, \alpha; J))$ .  $\square$

If the kernel of  $\alpha$  is complemented then the initial projection  $u^*u$  of the universal partial isometry  $u \in C^*(A, \alpha)^{**}$  can also be treated as a multiplier of  $A$ :

**Lemma 2.13.** *Suppose that  $(A, \alpha)$  is a  $C^*$ -dynamical system such that the kernel of  $\alpha$  is a complemented ideal in  $A$ . Let  $u \in C^*(A, \alpha)^{**}$  be the universal partial isometry. Then*

$$u^*uA = (\ker \alpha)^\perp \quad \text{and} \quad u^*\alpha(a)u = u^*ua, \quad a \in A.$$

*If  $(A, \alpha)$  is a reversible  $C^*$ -dynamical system then for every  $n \in \mathbb{N}$  the system  $(A, \alpha^n)$  is also reversible and*

$$(9) \quad u^{n*}u^nA = (\ker \alpha^n)^\perp \quad \text{and} \quad u^{n*}\alpha^n(a)u^n = u^{n*}u^na, \quad a \in A.$$

*Proof.* We have  $(\ker \alpha)^\perp = \{a \in A : u^*ua = a\} \subseteq u^*uA$  and  $u(\cdot)u^*$  maps both of the  $C^*$ -algebras  $(\ker \alpha)^\perp$  and  $u^*uA$  isomorphically onto  $\alpha(A) = uAu^*$ . This implies that  $(\ker \alpha)^\perp = u^*uA$ . In particular,  $u^*ua = u^*uau^*u = u^*\alpha(a)u$  for every  $a \in A$ .

Now assume that  $(A, \alpha)$  is reversible. Note that a composition of two homomorphisms  $f : A \rightarrow B$  and  $g : B \rightarrow C$  with hereditary ranges have a hereditary range. The latter holds because

$$\begin{aligned} g(f(A))Cg(f(A)) &= g(f(A))g(B)Cg(B)g(f(A)) = g(f(A))g(B)g(f(A)) \\ &= g(f(A)Bf(A)) = g(f(A)). \end{aligned}$$

Thus for every  $n \in \mathbb{N}$  the range of  $\alpha^n$  is a hereditary subalgebra of  $A$ . We prove that  $\ker \alpha^n$  is complemented and that (9) holds by induction on  $n$ . For  $n = 1$  we have already seen it. Assume that the assertion holds for some  $n \in \mathbb{N}$ . Let  $\theta$  be the inverse to the isomorphism  $\alpha^n : (\ker \alpha^n)^\perp \rightarrow \alpha^n(A)$ . Then clearly  $\theta(\alpha^n(A) \cap (\ker \alpha)^\perp) \subseteq (\ker \alpha^{n+1})^\perp$ . However, since  $\alpha^n(A)$  is hereditary in  $A$  we have  $\alpha^n(A) \cap (\ker \alpha)^\perp = \alpha^n(A)(\ker \alpha)^\perp \alpha^n(A)$ . Hence  $\alpha^{n+1}$  maps  $\theta(\alpha^n(A) \cap (\ker \alpha)^\perp)$  onto  $\alpha^{n+1}(A)$ . Since  $\alpha^{n+1}(\ker \alpha^{n+1})^\perp \rightarrow \alpha^{n+1}(A)$  is isometric it follows that it is actually an isomorphism and we have  $\theta(\alpha^n(A) \cap (\ker \alpha)^\perp) = (\ker \alpha^{n+1})^\perp$ . For any element  $\alpha^n(a)$  in  $(\ker \alpha)^\perp = u^*uA$ , by the induction hypothesis, we have

$$\theta(\alpha^n(a)) = u^{*n}\alpha^n(a)u^n = u^{*n}(u^*u)\alpha^n(a)u^n = u^{*n+1}u^{n+1}au^{*n}u^n = u^{*n+1}u^{n+1}a.$$

Hence  $(\ker \alpha^{n+1})^\perp \subseteq u^{*n+1}u^{n+1}A$ , and the argument we used to show that  $(\ker \alpha)^\perp = u^*uA$  implies that we actually have  $(\ker \alpha^{n+1})^\perp = u^{*n+1}u^{n+1}A$ . Thus  $u^{*n+1}u^{n+1} \in M(A)$  is the projection onto  $(\ker \alpha^{n+1})^\perp$ .  $\square$

Integration over the Haar measure on  $\mathbb{T}$  gives the (faithful) conditional expectation

$$(10) \quad \mathcal{E}(a) = \int_{\mathbb{T}} \gamma_z(a) d\mu, \quad a \in C^*(A, \alpha; J),$$

from  $C^*(A, \alpha; J)$  onto the fixed point  $C^*$ -algebra  $B$  for the gauge action. We refer to the  $C^*$ -algebra  $B$  as the *core  $C^*$ -subalgebra* of  $C^*(A, \alpha; J)$ . In view of Proposition 2.11, we have

$$B = \overline{\text{span}}\{u^{*n}au^n : a \in \alpha^n(A)A\alpha^n(A), n \in \mathbb{N}\}.$$

If the  $C^*$ -dynamical system  $(A, \alpha)$  is reversible, then the core of  $C^*(A, \alpha)$  coincides with  $A$  and  $C^*(A, \alpha)$  has a similar structure to that of classical crossed product by an automorphism.

**Proposition 2.14.** *Suppose that  $(A, \alpha)$  is a reversible  $C^*$ -dynamical system. The crossed product  $C^*(A, \alpha)$  is the closure of a dense  $*$ -algebra consisting of the elements of the form*

$$(11) \quad a = \sum_{k=1}^n u^{*k} a_{-k}^* + a_0 + \sum_{k=1}^n a_k u^k, \quad a_k \in A\alpha^k(A), \quad k = 0, \pm 1, \dots, \pm n.$$

The coefficients  $a_k \in A\alpha^k(A)$  in (11) are uniquely determined by  $a$ .

*Proof.* To see that any element (7) can be presented in the form (11) let us consider an element  $u^{*n}a_{n,m}u^m$  where  $a_{n,m} \in \alpha^n(A)A\alpha^m(A)$  and put  $k := m - n$ . Suppose that  $k \geq 0$ . Then  $\alpha^n(A)A\alpha^{n+k}(A) = \alpha^n(A)A\alpha^n(A)\alpha^{n+k}(A) = \alpha^n(A)\alpha^{n+k}(A) = \alpha^n(\alpha^k(A))$ . Thus there is  $a_k \in \alpha^k(A) \cap (\ker \alpha^k)^\perp$  such that  $a_{n,m} = \alpha^n(a_k)$ . Hence, by Lemma 2.13, we get

$$u^{*n}a_{n,m}u^m = u^{*n}\alpha^n(a_k)u^n u^k = a_k u^k.$$

If  $k < 0$  by passing to adjoints we get  $u^{*n}a_{n,m}u^m = u^{-*k}a_k$ . In view of Proposition 2.11, this proves the first part of the assertion. For the last part notice that if  $a$  is of the form (11), then for  $k \geq 0$  we have  $\mathcal{E}(u^k a) = a_{-k}^*$  and  $\mathcal{E}(a u^{*k}) = a_k$ . Hence the coefficients  $a_{\pm k}$  are uniquely determined by  $a$ .  $\square$

**2.3. Gauge-invariant ideals.** Let  $(A, \alpha)$  be a fixed  $C^*$ -dynamical system and let  $J$  be an ideal in  $(\ker \alpha)^\perp$ . Ideals in  $C^*(A, \alpha; J)$  that are invariant under the gauge action are called *gauge-invariant*. In this subsection we describe these ideals in terms of pairs of ideals in  $A$ .

**Definition 2.15** (Definitions 3.2 and 3.3 in [33]). We say that an ideal  $I$  in  $A$  is a *positively invariant* ideal in  $(A, \alpha)$  if  $\alpha(I) \subseteq I$ . We say that  $I$  is  *$J$ -negatively invariant* ideal in  $(A, \alpha)$  if  $J \cap \alpha^{-1}(I) \subseteq I$ . If  $I$  is both positively invariant and  $J$ -negatively invariant we say that  $I$  is  *$J$ -invariant*, and if  $J = (\ker \alpha)^\perp$  we drop the prefix ' $J$ '.

Let  $I$  be a positively invariant ideal in  $(A, \alpha)$ . It induces two  $C^*$ -dynamical systems: the *restricted  $C^*$ -dynamical system*  $(I, \alpha|_I)$  and the *quotient  $C^*$ -dynamical system*  $(A/I, \alpha_I)$  where  $\alpha_I(a + I) := \alpha(a) + I$  for all  $a \in A$ . Note that if  $(A, \alpha)$  is extendible then so is the quotient  $(A/I, \alpha_I)$ , but  $(I, \alpha|_I)$  in general fails to be extendible. For instance, if  $A = C_0(\mathbb{R})$ ,  $\alpha(a)(x) = a(x - 1)$  and  $I = C_0(0, \infty)$  then  $\alpha$  is extendible but  $\alpha|_I$  is not, see also [1].

**Lemma 2.16.** *Let  $I$  be an invariant ideal in  $(A, \alpha)$ .*

- (i) *If the kernel of  $\alpha$  is a complemented ideal in  $A$  then  $\alpha_I$  and  $\alpha|_I$  have complemented kernels in  $A/I$  and  $I$  respectively, and*

$$(\ker \alpha_I)^\perp = q_I((\ker \alpha)^\perp), \quad (\ker \alpha|_I)^\perp = (\ker \alpha)^\perp \cap I.$$

- (ii) *If  $(A, \alpha)$  is reversible then  $(A/I, \alpha_I)$  and  $(I, \alpha|_I)$  are reversible.*

*Proof.* (i). Since  $(\ker \alpha)^\perp \cap \alpha^{-1}(I) \subseteq I$  we get  $\ker \alpha_I = q_I(\alpha^{-1}(I)) = q_I((\ker \alpha)^\perp \cap \alpha^{-1}(I) + \ker \alpha) = q_I(\ker \alpha)$ . Thus  $q_I((\ker \alpha)^\perp) = (\ker \alpha_I)^\perp$  and  $q_I(\ker \alpha)$  is a complemented ideal in  $A/I$ . Since  $\ker \alpha|_I = \ker \alpha \cap I$  we see that  $(\ker \alpha|_I)^\perp = (\ker \alpha)^\perp \cap I$  and therefore these ideals are complementary in  $I$ .

(ii). In view of part (i) it suffices to note that both  $\alpha_I$  and  $\alpha|_I$  have hereditary ranges. The former is straightforward and the latter follows from the following relations

$$\alpha(I)I\alpha(I) = \alpha(IA)I\alpha(AI) = \alpha(I)\alpha(A)I\alpha(A)\alpha(I) \subseteq \alpha(I)\alpha(A)\alpha(I) = \alpha(I).$$

□

**Definition 2.17.** Let  $I, I', J$  be ideals in  $A$  where  $J \subseteq (\ker \alpha)^\perp$ . We say that  $(I, I')$  is a  $J$ -pair for a  $C^*$ -dynamical system  $(A, \alpha)$  if

$$I \text{ is positively invariant, } J \subseteq I' \text{ and } I' \cap \alpha^{-1}(I) = I.$$

The set of  $J$ -pairs for  $(A, \alpha)$  is equipped with a natural partial order induced by inclusion:  $(I_1, I'_1) \subseteq (I_2, I'_2) \stackrel{\text{def}}{\iff} I_1 \subseteq I_2 \text{ and } I'_1 \subseteq I'_2$ .

**Lemma 2.18.** If  $(\pi, U)$  is a  $J$ -covariant representation then  $(\ker \pi, I_{(\pi, U)})$  is a  $J$ -pair.

*Proof.* It is clear that  $\ker \pi$  is positively invariant,  $J \subseteq I_{(\pi, U)}$  and  $\ker \pi \subseteq I_{(\pi, U)}$ . In particular,  $\ker \pi \subseteq I_{(\pi, U)} \cap \alpha^{-1}(\ker \pi)$ . For the reverse inclusion, note that for any  $a \in I_{(\pi, U)} \cap \alpha^{-1}(\ker \pi)$  we have  $\pi(a) = U^*U\pi(a) = U^*U\pi(a)U^*U = U^*\pi(\alpha(a))U = 0$ . □

Clearly, if  $(I, I')$  is a  $J$ -pair, then  $I$  is  $J$ -invariant and  $I + J \subseteq I'$ . Note that then  $(I, I + J)$  is also a  $J$ -pair, but in general  $I + J \neq I'$ , cf. [33, Remark 3.2 and Example 3.1]. We have the following relationship between gauge-invariant ideals and  $J$ -pairs for  $(A, \alpha)$ .

**Theorem 2.19.** Let  $(A, \alpha)$  be a  $C^*$ -dynamical system and let  $J$  an ideal in  $(\ker \alpha)^\perp$ . The relations

$$(12) \quad I = A \cap \mathcal{I}, \quad I' = \{a \in A : (1 - u^*u)a \in \mathcal{I}\}$$

establish an order preserving bijective correspondence between  $J$ -pairs  $(I, I')$  for  $(A, \alpha)$  and gauge-invariant ideals  $\mathcal{I}$  in  $C^*(A, \alpha; J)$ . Moreover, for objects satisfying (12) we have a natural isomorphism

$$C^*(A, \alpha; J)/\mathcal{I} \cong C^*(A/I, \alpha_I; q_I(I'))$$

and if  $I' = I + J$  (equivalently  $\mathcal{I}$  is generated by its intersection with  $A$ ), then  $\mathcal{I}$  is Morita-Rieffel equivalent to  $C^*(I, \alpha|_I; I \cap J)$ .

*Proof.* We use Proposition A.8 to identify  $C^*(A, \alpha; J)$  with  $\mathcal{O}(J, E_\alpha)$ . By Theorem A.4 and Proposition A.10 the relations  $I = A \cap \mathcal{I}$  and  $I' = A \cap (\mathcal{I} + E_\alpha E_\alpha^*)$  establish a bijective correspondence between  $J$ -pairs  $(I, I')$  for  $(A, \alpha)$  and gauge-invariant ideals  $\mathcal{I}$  in  $C^*(A, \alpha; J)$ . Note that  $E_\alpha E_\alpha^* = u^* \alpha(A) A \alpha(A) u$  and recall that  $(1 - u^*u)$  is a multiplier of  $C^*(A, \alpha; J)$  by Corollary 2.12. Thus  $a \in I'$  implies that  $(1 - u^*u)a \in \mathcal{I}$ . Conversely, if  $a \in A$  is such that  $(1 - u^*u)a \in \mathcal{I}$  then, since  $(1 - u^*u)a = a - u^* \alpha(a) u$ , we have  $a \in \mathcal{I} + E_\alpha E_\alpha^*$ . Hence  $I' = \{a \in A : (1 - u^*u)a \in \mathcal{I}\}$ . Since  $E_{\alpha_I} \cong E_I$  we get  $C^*(A, \alpha; J)/\mathcal{I} \cong C^*(A/I, \alpha_I; q_I(I'))$ , see Theorem A.4 and Proposition A.10.

Clearly, the bijective correspondence  $(I, I') \longleftrightarrow \mathcal{I}$  preserves order. Thus  $\mathcal{I}$  is generated by  $I$  if and only if  $I' = I + J$ . In this case we see that  $\mathcal{I}$  is Morita-Rieffel equivalent to  $C^*(I, \alpha|_I; I \cap J)$  by Theorem A.4, because  $E_{\alpha|_I} \cong I E_\alpha$ . □

**Remark 2.20.** The pairs  $(\{0\}, J)$  and  $(\{0\}, (\ker \alpha)^\perp)$  are always  $J$ -pairs. Thus Theorem 2.19 implies that  $C^*(A, \alpha; J)$  is never simple unless  $J = (\ker \alpha)^\perp$ , that is unless  $C^*(A, \alpha; J) = C^*(A, \alpha)$ . More detailed necessary conditions and certain sufficient conditions for  $C^*(A, \alpha)$  to be simple can be found in [33, Theorem 4.2]. The only simplicity result we explicitly state in this paper is Proposition 5.9 below.

**Corollary 2.21.** *If the kernel of  $\alpha$  is a complemented ideal in  $A$  then the relations*

$$(13) \quad I = A \cap \mathcal{I}, \quad \mathcal{I} \text{ is generated by } I$$

*establish a bijective correspondence between invariant ideals  $I$  for  $(A, \alpha)$  and gauge-invariant ideals  $\mathcal{I}$  in  $C^*(A, \alpha)$ , under which we have  $C^*(A, \alpha)/\mathcal{I} \cong C^*(A/I, \alpha|_I)$  and  $\mathcal{I}$  is Morita-Rieffel equivalent to  $C^*(I, \alpha|_I)$ .*

*Proof.* Let  $(I, I')$  be a  $(\ker \alpha)^\perp$ -pair and let  $\mathcal{I}$  be the corresponding gauge-invariant ideal in  $C^*(A, \alpha)$ . Using (12) and Lemma 2.13 for any such pair we get

$$\begin{aligned} I' &= \{a \in A : (1 - u^*u)a \in \mathcal{I}\} = \{a \oplus b \in \ker \alpha \oplus (\ker \alpha)^\perp : a \in I\} \\ &= (\ker \alpha \cap I) \oplus (\ker \alpha)^\perp. \end{aligned}$$

Hence  $I' = I + (\ker \alpha)^\perp$ , that is  $\mathcal{I}$  is generated by  $I$ . By Lemma 2.16,  $\ker \alpha|_I = q_I(\ker \alpha)$  and  $(\ker \alpha)^\perp \cap I = (\ker \alpha|_I)^\perp$ . In particular, we get  $q_I(I') = q_I((\ker \alpha)^\perp) = (\ker \alpha|_I)^\perp$ . Now the assertion follows from Theorem 2.19.  $\square$

For crossed products of reversible  $C^*$ -dynamical systems we can actually identify gauge-invariant ideals up to isomorphism.

**Proposition 2.22.** *If  $(A, \alpha)$  is a reversible  $C^*$ -dynamical system and  $\mathcal{I}$  is a gauge-invariant ideal in  $C^*(A, \alpha)$ , then  $\mathcal{I} \cong C^*(I, \alpha|_I)$  where  $I = A \cap \mathcal{I}$ .*

*Proof.* In view of Propositions A.8 and A.11 it suffices to apply the general result for Hilbert bimodules [25, Theorem 10.6.]. Alternatively, the assertion can be proved directly using Lemma 2.16 and Propositions 2.14 and 2.9.  $\square$

**2.4. Extensions with complemented kernel.** We can use Corollary 2.21 to describe up to Morita-Rieffel equivalence all gauge-invariant ideals in an arbitrary crossed product  $C^*(A, \alpha; J)$ . More specifically, there is a canonical construction of a  $C^*$ -dynamical system  $(A_J, \alpha_J)$  such that  $C^*(A, \alpha; J) \cong C^*(A_J, \alpha_J)$  and the kernel of  $\alpha_J$  is complemented. The system  $(A_J, \alpha_J)$  was considered in [37, Subsection 6.1], in the case  $A$  is unital, but the construction works also in our general context.

**Definition 2.23.** For every  $C^*$ -dynamical system  $(A, \alpha)$  and an ideal  $J$  in  $(\ker \alpha)^\perp$  we put

$$A^J := (A/\ker \alpha) \oplus (A/J)$$

and define an endomorphism  $\alpha^J : A^J \rightarrow A^J$  by the formula

$$A^J \ni (a + \ker \alpha) \oplus (b + J) \xrightarrow{\alpha^J} (\alpha(a) + \ker \alpha) \oplus (\alpha(a) + J) \in A^J.$$

The system  $(A^J, \alpha^J)$  extends  $(A, \alpha)$ , in the sense that the map

$$A \ni a \xrightarrow{\iota^J} (a + \ker \alpha) \oplus (a + J) \in A^J$$

is an injective homomorphism that intertwines  $\alpha$  and  $\alpha^J$ . Moreover, the kernel of  $\alpha^J$  coincides with the direct summand  $A/J$  in  $A^J$ . Hence  $(\ker \alpha^J)^\perp$  corresponds to the  $A/\ker \alpha$  summand. We also note that  $\iota^J : A \rightarrow A^J$  is an isomorphism if and only if  $\ker \alpha$  is a complemented ideal in  $A$  and  $J = (\ker \alpha)^\perp$ .

**Lemma 2.24.** *If  $(I, I')$  is a  $J$ -pair for  $(A, \alpha)$  then*

$$(14) \quad (I, I')^J := q_{\ker \alpha}(I) \oplus q_J(I') \triangleleft A^J$$

*is an invariant ideal in  $(A^J, \alpha^J)$  such that*

$$(15) \quad I = \iota_J^{-1}((I, I')^J).$$

*Proof.* Let us prove (15) first. Since  $I \subseteq I'$  we have  $I \subseteq \iota_J^{-1}((I, I')^J)$ . If  $a \in \iota_J^{-1}((I, I')^J)$ , then  $a = i + k$  for some  $i \in I$ ,  $k \in \ker \alpha$ , and  $a \in I'$ . This implies that  $k \in I' \cap \ker \alpha \subseteq I' \cap \alpha^{-1}(I) = I$ . Hence  $a \in I$  and (15) holds.

Now using (15) and the equality  $\alpha^J(A^J) = \iota_J(\alpha(A))$  we get

$$\alpha^J(A^J) \cap (I, I')^J \subseteq \iota_J(I) \subseteq \alpha^J((I, I')^J).$$

On the other hand, since  $\alpha(I) \subseteq I \subseteq I'$  we have  $\alpha^J((I, I')^J) \subseteq (I, I')^J$ . Therefore  $\alpha^J((I, I')^J) = \alpha^J(A^J) \cap (I, I')^J$ . It follows that  $\alpha^J : (I, I')^J \cap (\ker \alpha^J)^\perp \rightarrow \alpha^J(A^J) \cap (I, I')^J$  is an isomorphism and this implies that  $(I, I')^J$  is invariant in  $C^*(A^J, \alpha^J)$ .  $\square$

**Proposition 2.25.** *Let  $(A, \alpha)$  be a  $C^*$ -dynamical system and let  $J$  be an ideal in  $(\ker \alpha)^\perp$ . The embedding  $\iota^J$  extends to a gauge-invariant isomorphism*

$$C^*(A, \alpha; J) \cong C^*(A^J, \alpha^J).$$

*If  $\mathcal{I}$  is a gauge-invariant ideal in  $C^*(A, \alpha; J)$  corresponding to a  $J$ -pair  $(I, I')$  for  $(A, \alpha)$ , then  $\mathcal{I}$  is mapped by the above isomorphism onto a gauge-invariant ideal in  $C^*(A^J, \alpha^J)$  which is generated by the ideal  $(I, I')^J$  given by (14). In particular,  $\mathcal{I}$  is Morita-Rieffel equivalent to  $C^*((I, I')^J, \alpha^J|_{(I, I')^J})$ .*

*Proof.* Let us denote by  $u$  and  $v$  the universal partial isometries in  $C^*(A, \alpha; J)$  and  $C^*(A_J, \alpha_J)$  respectively. It is clear that  $(\iota_J, v)$  is an injective representation of  $(A, \alpha)$  in  $C^*(A_J, \alpha_J)$  that admits a gauge action. Using Lemma 2.13 we get that  $\{a \in A : (v^*v)\iota_J(a) = \iota_J(a)\} = J$ . By virtue of Proposition 2.9 we see that  $\iota_J \rtimes v : C^*(A, \alpha; J) \rightarrow C^*(A_J, \alpha_J)$  is a gauge-invariant isomorphism. Note, again using Lemma 2.13, that for any  $a, b \in A$  we have

$$(1 - v^*v)((a + \ker \alpha) \oplus (b + J)) = 0 \oplus (b + J) = (1 - v^*v)\iota_J(b)$$

and

$$(v^*v)((a + \ker \alpha) \oplus (b + J)) = (a + \ker \alpha) \oplus 0 = (v^*v)\iota_J(a).$$

Let us now fix a  $J$ -pair  $(I, I')$  in  $(A, \alpha)$  and let  $(I, I')^J$  be the corresponding invariant ideal in  $(A^J, \alpha^J)$  given by (14). In view of the above equalities, we have

$$(16) \quad (I, I')^J = (v^*v)\iota_J(I) + (1 - v^*v)\iota_J(I').$$

Let  $\mathcal{I}^J$  be the ideal in  $C^*(A^J, \alpha^J)$  generated by  $(I, I')^J$  and let  $\mathcal{I} := (\iota_J \rtimes v)^{-1}(\mathcal{I}^J)$ . Then

$$\begin{aligned} \{a \in A : (1 - u^*u)a \in \mathcal{I}\} &= \{a \in A : (1 - v^*v)\iota_J(a) \in \mathcal{I}^J\} \\ &= \{a \in A : (1 - v^*v)\iota_J(a) \in (I, I')^J\} = I'. \end{aligned}$$



This, together with (15), shows that  $\mathcal{I}$  is the gauge-invariant ideal corresponding to the  $J$ -pair  $(I, I')$ . Hence  $\mathcal{I}$  is Morita-Rieffel equivalent to  $C^*((I, I')^J, \alpha^J|_{(I, I')^J})$ , by Corollary 2.21.  $\square$

**2.5.  $K$ -theory of gauge-invariant ideals.** We have the following generalization of the classical Pimsner-Voiculescu sequence.

**Proposition 2.26.** *For an ideal  $J$  in  $(\ker \alpha)^\perp$  we have the following exact sequence*

$$\begin{array}{ccccc} K_0(J) & \xrightarrow{K_0(\iota) - K_0(\alpha|_J)} & K_0(A) & \xrightarrow{K_0(\iota)} & K_0(C^*(A, \alpha; J)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(A, \alpha; J)) & \xleftarrow{K_1(\iota)} & K_1(A) & \xleftarrow{K_1(\iota) - K_1(\alpha|_J)} & K_1(J) \end{array}$$

where  $\iota$  stands for inclusion.

*Proof.* Using Lemma A.13 we see that in the sequence (48) we may replace the maps  $K_i(\iota_{22})^{-1} \circ K_i(\iota_{11} \circ \phi|_J)$  with  $K_i(\alpha|_J)$ ,  $i = 0, 1$ . This results with the desired sequence.  $\square$

One can combine results from previous subsections with Proposition 2.26 to get exact six-term sequences for  $K$ -theory of all gauge-invariant ideals and relevant quotients in the crossed product  $C^*(A, \alpha; J)$ . We state explicitly only results for gauge-invariant ideals.

**Theorem 2.27.** *Let  $\mathcal{I}$  be a gauge-invariant ideal in  $C^*(A, \alpha; J)$  where  $(A, \alpha)$  is a  $C^*$ -dynamical system and  $J$  is an ideal in  $(\ker \alpha)^\perp$ . Let  $(I, I')$  be the  $J$ -pair for  $(A, \alpha)$  given by (12). We have*

$$K_*(\mathcal{I}) \cong K_*(C^*((I, I')^J, \alpha^J|_{(I, I')^J})),$$

where  $(I, I')^J$  is given by (14), and in particular if  $K_1((I, I')^J) = 0$  then

$$(17) \quad K_0(\mathcal{I}) \cong \operatorname{coker}(K_0(\iota) - K_0(\alpha^J|_{q_{\ker \alpha}(I)})), \quad K_1(\mathcal{I}) \cong \ker(K_0(\iota) - K_0(\alpha^J|_{q_{\ker \alpha}(I)})),$$

where  $\alpha^J|_{q_{\ker \alpha}(I)} : q_{\ker \alpha}(I) \oplus \{0\} \rightarrow (I, I')^J$  is the restriction of  $\alpha^J$  and  $\iota : q_{\ker \alpha}(I) \rightarrow (I, I')^J$  is the inclusion. If  $\mathcal{I}$  is generated by  $I$ , that is if  $I' = I + J$ , then

$$K_*(\mathcal{I}) \cong K_*(C^*(I, \alpha|_I; I \cap J)),$$

and if additionally  $K_1(I) = K_1(I \cap J) = 0$ , then

$$K_0(\mathcal{I}) = \operatorname{coker}(K_0(\iota) - K_0(\alpha|_{I \cap J})), \quad K_1(\mathcal{I}) = \ker(K_0(\iota) - K_0(\alpha|_{I \cap J}))$$

where  $\alpha|_{I \cap J} : I \cap J \rightarrow I$  is the restriction of  $\alpha$  and  $\iota : I \cap J \rightarrow I$  is the inclusion.

*Proof.* By the last part of Proposition 2.25,  $\mathcal{I}$  is Morita-Rieffel equivalent to the crossed product  $C^*((I, I')^J, \alpha^J|_{(I, I')^J})$ . Hence the corresponding  $K$ -groups are isomorphic by [24, Proposition B.5], see also [24, Remark B.6]. If  $K_1((I, I')^J) = 0$  then also  $(\alpha^J|_{(I, I')^J})^\perp = q_{\ker \alpha}(I) \oplus \{0\}$  has  $K_1$ -group equal to zero. Thus applying Proposition 2.26 to the system  $((I, I')^J, \alpha^J|_{(I, I')^J})$  and the ideal  $(\ker \alpha^J|_{(I, I')^J})^\perp$  we get the second part of the assertion and (17). In view of the second part of Theorem 2.19, the above argument proves also the first part of the assertion.  $\square$

**Corollary 2.28.** *If  $\ker \alpha$  is a complemented ideal in  $A$  then for every gauge-invariant ideal  $\mathcal{I}$  in  $C^*(A, \alpha)$  we have*

$$K_*(\mathcal{I}) \cong K_*(C^*(I, \alpha|_I)), \quad \text{where } I := \mathcal{I} \cap A.$$

*If additionally  $K_1(I) = 0$ , then*

$$K_0(\mathcal{I}) = \text{coker}(K_0(\iota) - K_0(\alpha|_{I \cap (\ker \alpha)^\perp})), \quad K_1(\mathcal{I}) = \ker(K_1(\iota) - K_1(\alpha|_{I \cap (\ker \alpha)^\perp}))$$

*where  $\alpha|_{I \cap (\ker \alpha)^\perp} : I \cap (\ker \alpha)^\perp \rightarrow I$  is restriction of  $\alpha$  and  $\iota : I \cap (\ker \alpha)^\perp \rightarrow I$  is the inclusion.*

*Proof.* It suffices to combine the second parts of Theorem 2.27 and Corollary 2.21.  $\square$

**2.6. Reversible extensions.** We fix a  $C^*$ -dynamical system  $(A, \alpha)$  and an ideal  $J$  in  $(\ker \alpha)^\perp$ . We generalize a construction of a reversible  $C^*$ -dynamical system  $(B, \beta)$  associated to the triple  $(A, \alpha; J)$  in [33, Subsection 3.1], see also [36, Section 4], to the case when  $\alpha$  is not necessarily extendible. The system  $(B, \beta)$  can be viewed as a direct limit of approximating  $C^*$ -dynamical systems  $(B_n, \beta_n)$ ,  $n \in \mathbb{N}$ . We denote by  $q : A \rightarrow A/J$  the quotient map and for each  $n \in \mathbb{N}$  we put

$$A_n := \alpha^n(A)A\alpha^n(A).$$

The  $C^*$ -algebra  $B_n$ ,  $n \in \mathbb{N}$ , is a direct sum of the form

$$B_n = q(A_0) \oplus q(A_1) \oplus \dots \oplus q(A_{n-1}) \oplus A_n,$$

and the endomorphism  $\beta_n : B_n \rightarrow B_n$  is given by the formula

$$\beta_n(a_0 \oplus a_1 \oplus \dots \oplus a_n) = a_1 \oplus a_2 \oplus \dots \oplus q(a_n) \oplus \alpha(a_n),$$

where  $a_k \in q(A_k)$ ,  $k = 0, \dots, n-1$ , and  $a_n \in A_n$ ,  $n > 0$ . Thus we get a sequence  $(B_n, \beta_n)$ ,  $n \in \mathbb{N}$ , of  $C^*$ -dynamical systems where  $(B_0, \beta_0) = (A, \alpha)$ . We consider bonding homomorphisms  $\alpha_n : B_n \rightarrow B_{n+1}$ ,  $n \in \mathbb{N}$ , whose action is presented by the diagram

$$\begin{array}{ccccccc} B_n & = & q(A_0) & \oplus & \dots & \oplus & q(A_{n-1}) & \oplus & A_n & & . \\ \downarrow \alpha_n & & \downarrow id & & & & \downarrow id & & \downarrow q & \searrow \alpha & \\ B_{n+1} & = & q(A_0) & \oplus & \dots & \oplus & q(A_{n-1}) & \oplus & q(A_n) & \oplus & A_{n+1} \end{array}$$

In other words,  $\alpha_n$  is given by the formula

$$\alpha_n(a_0 \oplus \dots \oplus a_{n-1} \oplus a_n) = a_0 \oplus \dots \oplus a_{n-1} \oplus q(a_n) \oplus \alpha(a_n),$$

where  $a_k \in q(A_k)$ ,  $k = 0, \dots, n-1$ , and  $a_n \in A_n$ . Plainly, the homomorphisms  $\alpha_n$  are injective and we have

$$\alpha_n \circ \beta_n = \beta_{n+1} \circ \alpha_n, \quad n \in \mathbb{N}.$$

Accordingly, we get the direct sequence of  $C^*$ -dynamical systems:

$$(B_0, \beta_0) \xrightarrow{\alpha_0} (B_1, \beta_1) \xrightarrow{\alpha_1} (B_2, \beta_2) \xrightarrow{\alpha_2} \dots$$

We denote by  $(B, \beta)$  a direct limit of the above direct sequence. More precisely,  $B = \varinjlim \{B_n, \alpha_n\}$  is the  $C^*$ -algebraic direct limit, and  $\beta$  is determined by the formula  $\beta(\phi_n(a)) =$

$\phi_n(\beta_n(a))$  where  $\phi_n : B_n \rightarrow B$  is the natural (injective) homomorphism,  $a \in B_n$  and  $n \in \mathbb{N}$ . That is we have

$$(18) \quad \beta(\phi_n(a_0 \oplus a_1 \oplus \dots \oplus a_n)) = \phi_{n-1}(a_1 \oplus a_2 \oplus \dots \oplus a_n).$$

We now extend the main parts of [33, Theorem 3.1 and Proposition 4.7], see also [33, Remark 3.3].

**Theorem 2.29.** *The  $C^*$ -dynamical system  $(B, \beta)$  described above is reversible and we may assume a natural identification*

$$C^*(A, \alpha; J) = C^*(B, \beta)$$

under which we have

$$B = \overline{\text{span}}\{u^{*k}au^k : a \in \alpha^k(A)A\alpha^k(A), k \in \mathbb{N}\} \quad \text{and} \quad \beta(b) = ubu^*, \quad b \in B.$$

In particular, the relation  $\tilde{\pi}(\sum_{k=0}^n u^{*k}a_ku^k) = \sum_{k=0}^n U^{*k}\pi(a_k)U^k$ ,  $a_k \in \alpha^k(A)A\alpha^k(A)$ , establishes a one-to-one correspondence between  $J$ -covariant representations  $(\pi, U)$  of  $(A, \alpha)$  and covariant representations  $(\tilde{\pi}, U)$  of  $(B, \beta)$ .

*Proof.* Let us prove first that  $(B, \beta)$  is reversible. To this end, take  $a = a_0 \oplus a_1 \oplus \dots \oplus a_n \in B_n$  and  $b = b_0 \oplus b_1 \oplus \dots \oplus b_n \in B_{n-1}$  for  $n > 1$ . Then, in view of (18), we get

$$\beta(\phi_n(a))\phi_{n-1}(b)\beta(\phi_n(a)) = \beta(\phi_n(0 \oplus a_1b_0a_1 \oplus \dots \oplus a_nb_{n-1}a_n)).$$

This implies that  $\beta(B)B\beta(B) = \beta(B)$ . Hence  $\beta(B)$  is a hereditary  $C^*$ -subalgebra in  $B$ . The ideal  $\ker \beta$  is complemented in  $B$  as  $B_n \cap \ker \beta = \{\phi_n(a_0 \oplus 0 \oplus \dots \oplus 0) : a_0 \in q(A_0)\}$  is complemented in  $B_n$  for every  $n > 0$ . Thus  $(B, \beta)$  is reversible.

Now, for each  $n \in \mathbb{N}$ , we define  $C_n := \{\sum_{k=0}^n u^{*k}a_ku^k : a_k \in \alpha^k(A)A\alpha^k(A), k = 0, \dots, n\} \subseteq C^*(A, \alpha; J)$ . We also put  $C := \overline{\bigcup_{n \in \mathbb{N}} C_n}$ . It follows from (8) that  $C_n$ ,  $n \in \mathbb{N}$ , and  $C$  are  $C^*$ -algebras. Recall, see Proposition 2.11, that  $\{u^{*k}u^k\}_{k \in \mathbb{N}}$  is a decreasing sequence of orthogonal projections that commute with elements of  $A$ . Hence they commute with elements of  $C$ . Exactly as in the proof of [36, Statement 1], one checks that if  $a = \sum_{k=0}^n u^{*k}b_ku^k \in C_n$ ,  $b_k \in \alpha^k(A)A\alpha^k(A)$ ,  $k = 0, \dots, n$ , then putting  $a_k = \sum_{i=0}^k \alpha^{k-i}(b_i)$  we get that

$$(19) \quad a = \sum_{k=0}^{n-1} u^{*k}(1 - u^*u)a_ku^k + u^{*n}a_nu^n,$$

where 1 is the unit in  $M(C^*(A, \alpha; J))$ . Hence (19) is a general form of an element in  $C_n$ . In particular, since  $u^{*k}(1 - u^*u)u^k = (u^{*k}u^k - u^{*k+1}u^{k+1})$ ,  $k = 0, \dots, n-1$ , and  $u^{*n}u^n$  are mutually orthogonal projections commuting with elements of  $C_n$ , we see that  $C_n$  admits the following direct sum decomposition

$$C_n = \bigoplus_{k=0}^{n-1} (u^{*k}u^k - u^{*k+1}u^{k+1})C_n \oplus u^{*n}u^n C_n$$

Since  $u^k$  is a partial isometry it follows that

$$\alpha^k(A)C\alpha^k(A) = u^kAu^{*k}Cu^kAu^{*k} \ni a \rightarrow u^{*k}au^k \in C, \quad k = 1, \dots, n,$$

is a  $*$ -homomorphic isometry. Since  $J = u^*uA \cap A$  we also see, cf. for instance [21, Lemma 10.1.6], that

$$(1 - u^*u)A \ni a \rightarrow q(a) \in q(A)$$

is an isomorphism of  $C^*$ -algebras. Combining these facts we get that the formula

$$\Phi_n(q(a_0) \oplus q(a_1) \oplus \dots \oplus q(a_{n-1}) \oplus a_n) = \sum_{k=0}^{n-1} u^{*k}(1 - u^*u)a_k u^k + u^{*n}a_n u^n,$$

defines an isomorphism  $\Phi_n : B_n \rightarrow C_n$ . If  $a \in C_n$  is given by (19), then using equality  $u^{*n+1}\alpha(a_n)u^{n+1} = u^{*n}(u^*u)a_n u^n$  we get

$$a = \sum_{k=0}^n u^{*k}(1 - u^*u)a_k u^k + u^{*n+1}\alpha(a_n)u^{n+1}.$$

Therefore  $\Phi_{n+1} \circ \alpha_n = \Phi_n$ ,  $n \in \mathbb{N}$ . Hence the isomorphisms  $\Phi_n$  induce the isomorphism  $\Phi : B \rightarrow C$  between the inductive limit  $C^*$ -algebras  $B$  and  $C$ .

We claim that  $\Phi(\beta(b)) = ubu^*$ , for  $b \in C$ . Indeed, let  $a \in C_n$  is given by (19). Notice that for  $k > 0$  we have  $uu^{*k} = (uu^*)u^{*k-1} = u^{*k-1}u^{k-1}(uu^*)u^{*k-1} = u^{*k-1}u^k u^{*k}$ . Therefore, since  $u^k u^{*k} a_k = a_k$ , we get

$$u(u^{*k}(1 - u^*u)a_k u^k)u^* = u^{*k-1}(1 - u^*u)a_k u^{k-1}.$$

Clearly,  $u(1 - u^*u)a_0 u^* = ua_0 u^* - ua_0 u^* = 0$ . Accordingly,  $\Phi(\beta(\phi_n(a))) = u\Phi(\phi_n(a))u^*$ , which proves our claim.

It readily follows from the definition of  $\Phi_n$  that

$$u^*u\Phi_n(B_n) = \{\phi_n(0 \oplus a_1 \oplus \dots \oplus a_n) : 0 \oplus a_1 \oplus \dots \oplus a_n \in B_n\} = B_n \cap (\ker \beta)^\perp.$$

This implies that  $(\ker \beta)^\perp = \{b \in B : u^*u\Phi(b) = \Phi(b)\}$ .

Concluding the pair  $(\Phi, u)$  is an injective covariant representation of  $(B, \beta)$  in  $C^*(A, \alpha; J)$  that admits gauge-action. Thus, by Proposition 2.9,  $\Phi \rtimes u : C^*(B, \beta) \rightarrow C^*(A, \alpha; J)$  is an isomorphism which we may use to assumed the described identification. The last part of the assertion follows from the universal properties of the crossed products.  $\square$

**Definition 2.30** (Definition 3.1 in [33]). Suppose that  $(A, \alpha)$  is a  $C^*$ -dynamical system and  $J$  is an ideal in  $(\ker \alpha)^\perp$ . We call the  $C^*$ -dynamical system  $(B, \beta)$  constructed above the *natural reversible  $J$ -extension* of  $(A, \alpha)$ .

Let  $(B, \beta)$  be a natural reversible  $J$ -extension of  $(A, \alpha)$  and suppose that  $A = C_0(X)$  is commutative. Then, in view of our construction,  $B$  is also commutative and thus we may identify it with  $C_0(\tilde{X})$  where  $\tilde{X}$  is a locally compact Hausdorff space. With this identification,  $\beta$  is given by the formula

$$\beta(b)(\tilde{x}) = \begin{cases} b(\tilde{\varphi}(\tilde{x})), & \tilde{x} \in \tilde{\Delta}, \\ 0 & \tilde{x} \notin \tilde{\Delta}, \end{cases}$$

where  $\tilde{\varphi} : \tilde{\Delta} \rightarrow \tilde{\varphi}(\tilde{\Delta})$  is a homeomorphism,  $\tilde{\Delta} \subseteq \tilde{X}$  is open and  $\tilde{\varphi}(\tilde{\Delta}) \subseteq \tilde{X}$  is clopen. The pair  $(\tilde{X}, \tilde{\varphi})$  is uniquely determined by  $(X, \varphi)$  and the closed set

$$(20) \quad Y = \{x \in X : a(x) = 0 \text{ for all } a \in J\},$$

which necessarily contains  $X \setminus \varphi(\Delta)$ . Similarly as in [33, Proposition 4.7], cf. also [30, Theorem 3.5], using the above construction of  $(B, \beta)$  one can deduce the following description of  $(\tilde{X}, \tilde{\varphi})$ .

**Proposition 2.31.** *Up to conjugacy with a homeomorphism, the above partial dynamical system  $(\tilde{X}, \tilde{\varphi})$  can be described as follows:*

$$\tilde{X} = \bigcup_{N=0}^{\infty} X_N \cup X_{\infty}$$

where

$$X_N = \{(x_0, x_1, \dots, x_N, 0, \dots) : x_n \in \Delta, \varphi(x_n) = x_{n-1}, n = 1, \dots, N, x_N \in Y\},$$

$$X_{\infty} = \{(x_0, x_1, \dots) : x_n \in \Delta, \varphi(x_n) = x_{n-1}, n \geq 1\}.$$

The topology on  $\tilde{X}$  is the product one inherited from  $\prod_{n \in \mathbb{N}} (X \cup \{0\})$  where  $\{0\}$  is a clopen singleton and  $Y$  is given by (20). The homeomorphism  $\tilde{\varphi} : \tilde{\Delta} \rightarrow \tilde{\varphi}(\tilde{\Delta})$  is given by the formula

$$\tilde{\varphi}(x_0, x_1, \dots) = (\varphi(x_0), x_0, x_1, \dots), \quad \tilde{\Delta} = \{(x_0, x_1, \dots) \in \tilde{X} : x_0 \in \Delta\}.$$

*Proof.* We omit the proof as the assertion will follow from a much more general result we prove below, see Theorem 4.9.  $\square$

**Definition 2.32** (cf. Definition 3.5 [30]). Let  $Y$  be a closed subset of  $X$  that contains  $X \setminus \varphi(\Delta)$ . We call the dynamical system  $(\tilde{X}, \tilde{\varphi})$  described in the assertion of Proposition 2.31 the *natural reversible  $Y$ -extension of  $(X, \varphi)$* .

Note that for the natural reversible  $Y$ -extension  $(\tilde{X}, \tilde{\varphi})$  of  $(X, \varphi)$ , the map  $\Phi : \tilde{X} \rightarrow X$  given by  $\Phi(\tilde{x}) = x_0$  is surjective and intertwines  $\tilde{\varphi}$  and  $\varphi$ . This justifies the name.

**2.7. Topological freeness and freeness.** We turn to a discussion of certain conditions implying uniqueness property and gauge-invariance of all ideals in the crossed products. For reversible and extendible systems the relevant statements in [33, Subsection 4.5] were deduced from [34, Theorem 2.20]. We will extend them by applying general results from [32] and facts presented in Appendix A.

**Definition 2.33.** Let  $\varphi$  be a partial homeomorphism of a topological (not necessarily Hausdorff) space  $X$  with domain being an open set  $\Delta \subseteq X$ . We say that  $\varphi$  is *topologically free* if the set of its periodic points of any given period  $n > 0$  has empty interior. A set  $V \subseteq X$  is *invariant* if  $\varphi(V \cap \Delta) = V \cap \varphi(\Delta)$ . We say that  $\varphi$  is (essentially) *free*, if it is topologically free when restricted to any closed invariant set.

**Definition 2.34.** Let  $(A, \alpha)$  be a reversible  $C^*$ -dynamical system. Since  $(\ker \alpha)^{\perp}$  is an ideal in  $A$  and  $\alpha(A) = \alpha(A)A\alpha(A)$  is a hereditary subalgebra of  $A$  we have the natural identifications:

$$\widehat{(\ker \alpha)^{\perp}} = \{\pi \in \hat{A} : \pi((\ker \alpha)^{\perp}) \neq 0\}, \quad \widehat{\alpha(A)} = \{\pi \in \hat{A} : \pi(\alpha(A)) \neq 0\}.$$

Thus we treat  $\widehat{\alpha(A)}$  and  $\widehat{(\ker \alpha)^{\perp}}$  as open subsets of  $\hat{A}$ . With these identifications the homeomorphism  $\hat{\alpha} : \widehat{\alpha(A)} \rightarrow \widehat{(\ker \alpha)^{\perp}}$  dual to the isomorphism  $\alpha : (\ker \alpha)^{\perp} \rightarrow \alpha(A)$  becomes a partial homeomorphism of the spectrum of  $\hat{A}$ , cf. [33]. We refer to  $\hat{\alpha}$  as to the *partial homeomorphism dual to  $(A, \alpha)$* .

**Proposition 2.35.** *Let  $(A, \alpha)$  be a reversible  $C^*$ -dynamical system.*

- (i) If  $\hat{\alpha}$  is topologically free, then every injective covariant representation  $(\pi, U)$  of  $(A, \alpha)$  give rise to a faithful representation of  $C^*(A, \alpha)$ .
- (ii) If  $\hat{\alpha}$  is free, then all ideals in  $C^*(A, \alpha)$  are gauge-invariant; hence they are in one-to-one correspondence with invariant ideals in  $(A, \alpha)$ , cf. Corollary 2.21.

*Proof.* By Proposition A.8 and Lemma A.12, Theorem A.6 translates to the desired assertion.  $\square$

One can apply the above proposition to an arbitrary crossed product  $C^*(A, \alpha; J)$  using the identification  $C^*(A, \alpha; J) = C^*(B, \beta)$  from Theorem 2.29, and the following lemma.

**Lemma 2.36.** *Let  $(A, \alpha)$  be a  $C^*$ -dynamical system,  $J$  an ideal in  $(\ker \alpha)^\perp$ , and  $(B, \beta)$  the natural reversible  $J$ -extension of  $(A, \alpha)$ .*

- (i) *For any injective  $J$ -covariant representation  $(\pi, U)$  of  $(A, \alpha)$  the corresponding covariant representation  $(\tilde{\pi}, U)$  of  $(B, \beta)$  is injective if and only if  $J = \{a \in A : U^*U\pi(a) = \pi(a)\}$ .*
- (ii) *Relations (12) establish a bijective correspondence between with  $J$ -pairs  $(I, I')$  in  $(A, \alpha)$  and invariant ideals in  $(B, \beta)$ .*

*Proof.* (ii). Since  $C^*(A, \alpha; J) = C^*(B, \beta)$  we get the assertion by applying Theorem 2.19 to  $C^*(A, \alpha; J)$  and Corollary 2.21 to  $C^*(B, \beta)$ .

(i). It follows from item (ii) and Lemma 2.18.  $\square$

In practice, in order to use Proposition 2.35 and Lemma 2.36, one has to determine topological freeness and freeness of  $\hat{\beta}$  in terms of  $(A, \alpha)$  and  $J$ . This can be readily achieved if  $A$  is commutative.

**Definition 2.37** (Definition 4.8 in [33]). Let  $\varphi$  be a partial mapping of a locally compact Hausdorff space  $X$  defined on an open set  $\Delta \subseteq X$ . We say that a periodic orbit  $\mathcal{O} = \{x, \varphi(x), \dots, \varphi^{n-1}(x)\}$  of a periodic point  $x = \varphi^n(x)$  has an entrance  $y \in \Delta$  if  $y \notin \mathcal{O}$  and  $\varphi(y) \in \mathcal{O}$ . We say  $\varphi$  is *topologically free outside a set*  $Y \subseteq X$  if the set of periodic points whose orbits do not intersect  $Y$  and have no entrances have empty interior.

**Lemma 2.38.** *Let  $(\tilde{X}, \tilde{\varphi})$  be the  $Y$ -extension of a partial dynamical system  $(X, \varphi)$  where  $Y$  is a closed set containing  $X \setminus \varphi(\Delta)$ , see Definition 2.32. Then*

- (i)  *$\tilde{\varphi}$  is topologically free if and only if  $\varphi$  is topologically free outside  $Y$ ,*
- (ii)  *$\tilde{\varphi}$  is free if and only if  $\varphi$  is free (has no periodic points).*

*Proof.* Item (i) can be proved exactly as [33, Lemma 4.2]. Item (ii) is straightforward.  $\square$

One of the aims of the present paper is to obtain effective conditions implying the properties of crossed products described in Proposition 2.35 for a class of  $C^*$ -dynamical systems on  $C_0(X)$ -algebras. This is achieved in Theorems 4.11 and 4.12 below.

**2.8. Pure infinite crossed products for reversible  $C^*$ -dynamical systems.** In this subsection, we fix a reversible  $C^*$ -dynamical system  $(A, \alpha)$ . The property that we are about to introduce appears (without a name) in a number of proofs of pure infiniteness for crossed products. As we explain in more detail below, in the context of crossed products, this property is formally weaker than spectral freeness [46], topological freeness, proper

outererness [10] and aperiodicity [38], but the general relationship between these notions is not completely clear.

**Definition 2.39.** Let  $A$  be a  $C^*$ -subalgebra of a  $C^*$ -algebra  $B$ . We say that  $A^+$  *supports elements* of  $B^+$  if for every  $b \in B^+ \setminus \{0\}$  there exists  $a \in A^+$  such that  $a \preceq b$ . We say that  $A^+$  *residually supports elements* of  $B^+$  if for every ideal  $I$  of  $B$ ,  $q_I(A)^+$  supports elements of  $q_I(B)$ .

**Remark 2.40.** If  $A^+$  is a filling family for  $B$  in the sense of [28, Definition 4.2] then  $A^+$  residually supports elements of  $B^+$  (it is not clear whether the converse implication holds). Thus, if  $A$  is commutative or separable and  $\alpha : A \rightarrow A$  is a residually properly outer automorphism, then [29, Theorem 3.8] implies that  $A^+$  residually supports elements of  $C^*(A, \alpha)^+$ . In [46, Proposition 3.9] it is shown that  $A^+$  supports elements of  $B^+$  if and only if for every  $b \in B^+ \setminus \{0\}$  there is  $z \in B$  such that  $zaz^*$  is a non-zero element of  $A$ . In particular, [46, Lemma 3.2] implies that if  $\alpha : A \rightarrow A$  is an automorphism and the corresponding  $\mathbb{Z}$ -action is spectrally free in the sense of [46, Definition 1.3], then  $A^+$  residually supports elements of  $C^*(A, \alpha)^+$ .

In connection with Remark 2.40 we show that the notion of residual aperiodicity introduced in [38, Definition 8.19], for (a semigroup version of) extendible reversible systems, implies that  $A^+$  residually supports elements of  $C^*(A, \alpha)^+$ .

**Definition 2.41.** We say that an extendible reversible  $C^*$ -dynamical system  $(A, \alpha)$  is *aperiodic* if for each  $n > 0$ , each  $a \in A$  and every hereditary subalgebra  $D$  of  $A$

$$\inf\{\|da\alpha^n(d)\| : d \in D^+, \|d\| = 1\} = 0.$$

We say that  $(A, \alpha)$  is *residually aperiodic* if the quotient system  $(A/I, \alpha_I)$  is aperiodic for every invariant ideal  $I$  in  $(A, \alpha)$ .

**Proposition 2.42.** *If  $(A, \alpha)$  is residually aperiodic then  $A^+$  residually supports elements of  $C^*(A, \alpha)^+$ .*

*Proof.* By [38, Lemmas 8.18], [38, Corollary 4.7] every ideal in  $C^*(A, \alpha)$  is generated by its intersection with  $A$ . Let  $\mathcal{I}$  be an ideal in  $C^*(A, \alpha)$ . By Corollary 2.21, we have the isomorphism  $C^*(A, \alpha)/\mathcal{I} \cong C^*(A/I, \alpha_I)$  where  $I := A \cap \mathcal{I}$  is an invariant ideal in  $(A, \alpha)$ . The system  $(A/I, \alpha_I)$  is reversible by Lemma 2.16 ii). Fix a positive element  $b$  in  $C^*(A, \alpha)/\mathcal{I}$ . We may assume that  $\|b\| = 1$ . Applying to  $(A/I, \alpha_I)$  [38, Lemmas 4.2 and 8.18], we may find a positive contraction  $h \in A/I$  such that

$$(21) \quad \|h\mathcal{E}(b)h - hbh\| \leq 1/4, \quad \|h\mathcal{E}(b)h\| \geq \|\mathcal{E}(b)\| - 1/4 = 3/4$$

where  $\mathcal{E}$  is the conditional expectation from  $C^*(A/I, \alpha_I)$  onto  $A/I$ . Putting  $a := (h\mathcal{E}(b)h - 1/2)_+ \in A/I$  we have that  $a \neq 0$  because  $\|h\mathcal{E}(b)h\| > 1/2$ . Moreover, by [27, Proposition 2.2], relations  $\|h\mathcal{E}(b)h\| > 1/2$  and  $\|h\mathcal{E}(b)h - hbh\| \leq 1/4$  imply that  $a \preceq hbh$  relative to  $C^*(A/I, \alpha_I) \cong C^*(A, \alpha)/\mathcal{I}$ .  $\square$

Before we prove the main result of this subsection we need two lemmas.

**Lemma 2.43.** *If  $A^+$  residually supports elements of  $C^*(A, \alpha)^+$  then every ideal in  $C^*(A, \alpha)$  is gauge-invariant.*

*Proof.* Let  $\mathcal{I}$  be an ideal in  $C^*(A, \alpha)$  and let  $\langle I \rangle$  be the smallest ideal in  $C^*(A, \alpha)$  containing  $I := \mathcal{I} \cap A$ . By Corollary 2.21, we may identify  $C^*(A, \alpha)/\langle I \rangle$  with  $C^*(A/I, \alpha_I)$ . We have a natural epimorphism  $\Phi : C^*(A, \alpha)/\langle I \rangle \rightarrow C^*(A, \alpha)/\mathcal{I}$  which is injective on  $A/I$ . For any non-zero positive element  $b$  in  $C^*(A, \alpha)/\langle I \rangle$  there is a non-zero positive element  $a$  in  $A/I$  such that  $a \lesssim b$ . Since  $0 \neq \Phi(a) \lesssim \Phi(b)$ , we conclude that  $\Phi(b) \neq 0$ . Thus  $\ker \Phi = \{0\}$  and therefore  $\mathcal{I} = \langle I \rangle$  is gauge-invariant.  $\square$

**Lemma 2.44.** *Let  $A \subseteq B$  be  $C^*$ -algebras and let  $A$  be of real rank zero. The following conditions are equivalent*

- (i) *Every non-zero positive element in  $A$  is properly infinite in  $B$ .*
- (ii) *Every non-zero projection in  $A$  is properly infinite in  $B$ .*

*Proof.* Implication (i) $\Rightarrow$ (ii) is trivial. Assume that (ii) holds and let  $a \in A$  be a non-zero positive element. By [8, Theorem 2.6] there is an approximate unit  $\{p_\lambda : \lambda \in \Lambda\}$  in  $\overline{aAa}$  consisting of projections. Thus, by [27, Proposition 2.7(i)],  $p_\lambda \lesssim a$  for all  $\lambda$ , in  $A$  and all the more in  $B$ . Applying [27, Lemma 3.17(ii)] we see that  $\{p_\lambda : \lambda \in \Lambda\} \subseteq J(a) := \{x \in B : a \oplus |x| \lesssim a\}$ . Thus  $B\{p_\lambda : \lambda \in \Lambda\}B \subseteq J(a)$  because  $J(a)$  is an ideal, see [27, Lemma 3.12(i)]. On the other hand  $J(a) \subseteq BaB$  by [27, Lemma 3.12(iii)] and since we clearly have  $BaB \subseteq B\{p_\lambda : \lambda \in \Lambda\}B$  it follows that  $J(a) = BaB$ . Hence [27, Lemma 3.12(iv)] tells us that  $a$  is properly infinite in  $B$ .  $\square$

**Remark 2.45.** The equivalence of (i) and (ii) in Lemma 2.44 answers the question posed in the proof of [16, Theorem 4.4]: it shows that [16, Theorem 4.4] can be deduced from [16, Theorem 4.2].

**Proposition 2.46** (pure infiniteness criterion). *Let  $(A, \alpha)$  be a reversible  $C^*$ -dynamical system such that  $A^+$  residually supports elements of  $C^*(A, \alpha)^+$ . Suppose also that either  $A$  has the ideal property or that  $A$  is separable and there finitely many invariant ideals in  $(A, \alpha)$ . The following statements are equivalent:*

- (i) *Every non-zero positive element in  $A$  is properly infinite in  $C^*(A, \alpha)$ .*
- (ii)  *$C^*(A, \alpha)$  is purely infinite.*
- (iii)  *$C^*(A, \alpha)$  is purely infinite and has the ideal property.*
- (iv) *Every non-zero hereditary  $C^*$ -subalgebra in any quotient  $C^*(A, \alpha)$  contains an infinite projection.*

*If  $A$  is of real rank zero, then each of the above conditions is equivalent to*

- (i') *Every non-zero projection in  $A$  is properly infinite in  $C^*(A, \alpha)$ .*

*In particular, if  $A$  is purely infinite then  $C^*(A, \alpha)$  is purely infinite and has the ideal property.*

*Proof.* Implications (iv) $\Leftrightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are general facts, see respectively [47, Propositions 2.11], [27, Proposition 4.7] and [27, Theorem 4.16]. If  $A$  is of real rank zero the equivalence (i) $\Leftrightarrow$ (i') is ensured by Lemma 2.44. Thus it suffices to show that (i) implies (iii) or (iv). Let us then assume that every element in  $A^+ \setminus \{0\}$  is properly infinite in  $C^*(A, \alpha)$ .

Suppose first that  $A$  has the ideal property. We will show (iv). Let  $\mathcal{I}$  be an ideal in  $C^*(A, \alpha)$  and let  $B$  be a non-zero hereditary  $C^*$ -subalgebra in the quotient  $C^*(A, \alpha)/\mathcal{I}$ . Fix a non-zero positive element  $b$  in  $B$ . Since  $A^+$  residually supports elements of  $C^*(A, \alpha)^+$  is non-zero positive element  $a$  in  $q_{\mathcal{I}}(A)$  such that  $a \lesssim b$ . Note that  $a$  is properly infinite



in  $C^*(A, \alpha)/\mathcal{I}$  by [27, Proposition 3.14]. Since  $A$  has the ideal property we can find a projection  $q \in A$  that belongs to the ideal in  $A$  generated by the preimage of  $a$  in  $A$  but not to  $I := A \cap I$ . Then  $q + \mathcal{I}$  belongs to the ideal in  $C^*(A, \alpha)/\mathcal{I}$  generated by  $a$ , whence  $q + \mathcal{I} \lesssim a \lesssim b$ , by [27, Proposition 3.5(ii)]. From the comment after [27, Proposition 2.6] we can find  $z \in C^*(A, \alpha)/\mathcal{I}$  such that  $q + \mathcal{I} = z^*bz$ . With  $v := b^{\frac{1}{2}}z$  it follows that  $v^*v = q + \mathcal{I}$ , whence  $p := vv^* = b^{\frac{1}{2}}zz^*b^{\frac{1}{2}}$  is a projection in  $B$ , which is equivalent to  $q + \mathcal{I}$ . By our assumption  $q + \mathcal{I}$  and hence also  $p$  is properly infinite.

Suppose now that  $A$  is separable and there are finitely many, say  $n$ , invariant ideals in  $(A, \alpha)$ . By Lemma 2.43 and Corollary 2.21 they are in one-to-one correspondence with ideals in  $C^*(A, \alpha)$ . Hence by [27, Proposition 2.11], the conditions (ii) and (iii) are equivalent. We will prove (ii). The proof goes by induction on  $n$ .

Assume first that  $n = 2$  so that  $C^*(A, \alpha)$  is simple. For any  $b \in C^*(A, \alpha)^+ \setminus \{0\}$  take  $a \in A^+ \setminus \{0\}$  such that  $a \lesssim b$ . Then  $b \in C^*(A, \alpha)aC^*(A, \alpha) = C^*(A, \alpha)$  and as  $a$  is properly infinite we get  $b \lesssim a$  by [27, Proposition 3.5]. Hence  $b$  is properly infinite as it is Cuntz equivalent to  $a$ . Thus  $C^*(A, \alpha)$  purely infinite.

Now suppose that our claim holds for any  $k < n$ . Let  $\mathcal{I}$  be any non-trivial ideal in  $C^*(A, \alpha)$  and put  $I = \mathcal{I} \cap A$ . By Lemma 2.16 ii), the systems  $(A/I, \alpha_I)$  and  $(I, \alpha|_I)$  are reversible, and by Corollary 2.21 and Proposition 2.22 we have  $C^*(A/I, \alpha_I) \cong C^*(A, \alpha)/\mathcal{I}$  and  $C^*(I, \alpha|_I) \cong \mathcal{I}$ . Clearly, the system  $(A/I, \alpha_I)$  satisfies the assumptions of the assertion (a non-zero image of properly infinite element is properly infinite by [27, Proposition 3.14]) and there are less than  $n$  invariant ideals in  $(A/I, \alpha_I)$ . Hence  $C^*(A/I, \alpha_I)$  is purely infinite. Similar argument works for  $C^*(I, \alpha|_I)$ ; in particular note that if  $a \lesssim b$  for  $b \in \mathcal{I}^+ \setminus \{0\}$  and  $a \in A^+ \setminus \{0\}$ , then  $a \in I$ . Also if  $a \in I^+ \setminus \{0\}$  is properly infinite in  $C^*(A, \alpha)$ , then it is properly infinite in  $\mathcal{I}$ , cf. [27, Proposition 3.3]. Concluding, both  $\mathcal{I}$  and  $C^*(A, \alpha)/\mathcal{I}$  are purely infinite, and since pure infiniteness is closed under extensions [27, Theorem 4.19] we get that  $C^*(A, \alpha)$  is purely infinite.  $\square$

**Remark 2.47.** We recall, see [47, Propositions 2.11, 2.14], that in the presence of the ideal property pure infiniteness of a  $C^*$ -algebra is equivalent to strong pure infiniteness, weak pure infiniteness, and many other notions of infiniteness appearing in the literature. Thus the list of equivalent conditions in Proposition 2.46 can be considerably extended.

**Remark 2.48.** In the case when there are finitely many invariant ideals in  $(A, \alpha)$ , we used separability of  $A$  in the proof Proposition 2.46 only to get the equivalence (ii)  $\Leftrightarrow$  (iii). Accordingly, in this case, the conditions (i) and (ii) are equivalent even for non-separable  $C^*$ -algebras.

**Remark 2.49.** Certain properties that imply condition (i) in Proposition 2.46 were introduced in [38]. In particular, Proposition 2.46 can be readily used to obtain a generalization of [38, Theorem 8.22] so that it covers not necessarily extendible systems on  $C^*$ -algebras not necessarily possessing the ideal property.

### 3. CATEGORY OF $C_0(X)$ -ALGEBRAS AND $C_0(X)$ -DYNAMICAL SYSTEMS

In this section, we introduce morphisms of upper semicontinuous  $C^*$ -bundles which induce certain homomorphisms of  $C_0(X)$ -algebras. We give several characterizations of such homomorphisms, and study basic properties of  $C^*$ -dynamical systems  $(A, \alpha)$  where  $A$  is

a  $C_0(X)$ -algebra and  $\alpha$  is induced by a morphism. We show that the arising category of  $C_0(X)$ -algebras has direct limits, and in some cases such limits exist in the subcategory of continuous  $C_0(X)$ -algebras.

**3.1. Morphism of  $C^*$ -bundles and  $C_0(X)$ -dynamical systems.** Let  $\mathcal{A} = \bigsqcup_{x \in X} A(x)$  and  $\mathcal{B} = \bigsqcup_{y \in Y} B(y)$  be upper semicontinuous  $C^*$ -bundles. We wish to view morphism between  $C^*$ -bundles as a common generalization of proper mappings and  $C^*$ -homomorphisms. Mimicking the definition of morphisms of vector bundles, one can imagine such a morphism as a pair of continuous mappings  $\alpha : \mathcal{B} \rightarrow \mathcal{A}$  and  $\varphi : X \rightarrow Y$  such that the following diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\alpha} & \mathcal{A} \\ p \downarrow & & \downarrow p \\ Y & \xleftarrow{\varphi} & X \end{array}$$

commutes and for each  $x \in X$ ,  $\alpha : B(\varphi(x)) \rightarrow A(x)$  is a homomorphism. Since some of these homomorphisms might be zero we will allow  $\varphi$  to be defined on an open subset  $\Delta$  of  $X$ .

**Definition 3.1.** A *morphism* (of upper semicontinuous  $C^*$ -bundles) from  $\mathcal{B}$  to  $\mathcal{A}$  is a pair  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  consisting of

- 1) a continuous proper map  $\varphi : \Delta \rightarrow Y$  defined on an open set  $\Delta \subseteq X$ , and
- 2) a continuous bundle of homomorphisms  $\{\alpha_x\}_{x \in \Delta}$  between the corresponding fibers, i.e.:
  - a) for each  $x \in \Delta$ ,  $\alpha_x : B(\varphi(x)) \rightarrow A(x)$  is a homomorphism;
  - b) if  $\{x_i\}_{i \in \Lambda} \subseteq \Delta$  and  $\{b_i\}_{i \in \Lambda} \subseteq \mathcal{B}$  are nets such that  $x_i \rightarrow x \in \Delta$ ,  $b_i \rightarrow b$  and  $p(b_i) = \varphi(x_i)$ , for  $i \in \Lambda$ , then  $\alpha_{x_i}(b_i) \rightarrow \alpha_x(b)$ .

The above definition is born to work well with section algebras.

**Proposition 3.2.** Let  $\varphi : \Delta \rightarrow Y$  be a proper continuous mapping where  $\Delta \subseteq X$  is an open set. For each  $x \in \Delta$  let  $\alpha_x : B(\varphi(x)) \rightarrow A(x)$  be a homomorphism. The pair  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  is a morphism from  $\mathcal{B}$  to  $\mathcal{A}$  if and only if the formula

$$(22) \quad \alpha(b)(x) = \begin{cases} \alpha_x(b(\varphi(x))), & x \in \Delta, \\ 0_x & x \notin \Delta, \end{cases} \quad b \in \Gamma_0(\mathcal{B}), \quad x \in X,$$

yields a well defined homomorphism  $\alpha : \Gamma_0(\mathcal{B}) \rightarrow \Gamma_0(\mathcal{A})$  between the section  $C^*$ -algebras.

*Proof.* Suppose that  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  is a morphism. Clearly, it suffices to show that the map (22) is well defined, equivalently, that for any  $b \in \Gamma_0(\mathcal{B})$  the mapping

$$(23) \quad X \ni x \mapsto \alpha(b)(x) \in A(x) \subseteq \mathcal{A}$$

is in  $\Gamma_0(\mathcal{A})$ . Condition 2b) from Definition 3.1 readily implies that the map  $\Delta \ni x \mapsto \alpha(b)(x) \in A(x) \subseteq \mathcal{A}$  is continuous (consider elements  $b_i := b(\varphi(x_i))$ ). In particular,  $\Delta \ni x \mapsto \|\alpha(b)(x)\| \in \mathbb{R}$  is upper semicontinuous. Thus for any  $\varepsilon > 0$  the set  $\{x \in X : \|\alpha(b)(x)\| \geq \varepsilon\} = \{x \in \Delta : \|\alpha(b)(x)\| \geq \varepsilon\}$  is closed. Actually it is compact because

$$\{x \in X : \|\alpha(b)(x)\| \geq \varepsilon\} \subseteq \{x \in \Delta : \|b(\varphi(x))\| \geq \varepsilon\}$$

and the latter set is compact as  $\varphi$  is proper and  $b$  vanishes at infinity. Thus the map (23) is vanishing at infinity. To conclude that  $\alpha(b) \in \Gamma_0(\mathcal{A})$  we need to show that  $\alpha(b)$  is continuous on the boundary  $\partial\Delta$  of  $\Delta$ . But if  $\{x_i\}_{i \in \Lambda} \subseteq X$  is a net convergent to  $x_0 \in \partial\Delta$ , then for every  $\varepsilon > 0$  the point  $x_0$  belongs to the open set  $\{x \in X : \|\alpha(b)(x)\| < \varepsilon\}$  and hence  $\alpha(b)(x_i)$  converges to 0 by Lemma 1.1 (consider  $b_i = \alpha(b)(x_i)$  and  $a \equiv 0$ ).

Conversely, assume that  $\alpha : \Gamma_0(\mathcal{B}) \rightarrow \Gamma_0(\mathcal{A})$  is a homomorphism satisfying (22). We need to show condition 2b) in Definition 3.1. Let  $\{x_i\}_{i \in \Lambda} \subseteq \Delta$  and  $\{b_i\}_{i \in \Lambda} \subseteq \mathcal{B}$  be nets such that  $x_i \rightarrow x \in \Delta$ ,  $b_i \rightarrow b$  and  $p(b_i) = \varphi(x_i)$ . Take arbitrary  $\varepsilon > 0$ . By Lemma 1.1 there is  $a \in \Gamma_0(\mathcal{A})$  such that  $\|a(p(b)) - b\| < \varepsilon$  and we eventually have  $\|a(\varphi(x_i)) - b_i\| < \varepsilon$ . This implies that  $\|\alpha(a)(x) - \alpha_x(b)\| < \varepsilon$  and we eventually have  $\|\alpha(a)(x_i) - \alpha_{x_i}(b_i)\| < \varepsilon$ . Since  $\alpha(a) \in \Gamma_0(\mathcal{A})$  we have  $\alpha_{x_i}(b_i) \rightarrow \alpha_x(b)$  by Lemma 1.1.  $\square$

**Definition 3.3.** To indicate that a homomorphism  $\alpha : \Gamma_0(\mathcal{B}) \rightarrow \Gamma_0(\mathcal{A})$  is given by (22) for a certain morphism  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  of upper semicontinuous  $C^*$ -bundles we will say that  $\alpha$  is *induced by a morphism*.

Let  $A = \Gamma_0(\mathcal{A})$  and  $B = \Gamma_0(\mathcal{B})$ . Note that for an induced homomorphism  $\alpha : B \rightarrow A$  the underlying mapping  $\varphi : \Delta \rightarrow Y$  is uniquely determined by  $\alpha$  on the set

$$(24) \quad \Delta_0 := \{x \in X : \alpha_x \neq 0\} \subseteq \Delta,$$

which coincides with  $\Delta$  when all endomorphisms  $\alpha_x$ ,  $x \in \Delta$ , are non-zero. Sometimes we can assume that  $\Delta = \Delta_0$  using the following lemma.

**Lemma 3.4.** *Let  $\alpha : B \rightarrow A$  be a homomorphism induced by a morphism  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  from a continuous  $C_0(Y)$ -algebra  $B$  to a continuous  $C_0(X)$ -algebra  $A$  and let  $\Delta_0$  be given by (24). Suppose also that  $B(\varphi(x)) \neq \{0\}$ , for  $x \in \Delta$ , and that every  $\alpha_x$ ,  $x \in \Delta_0$ , is injective. Then  $\Delta_0$  is a clopen in  $\Delta$  and in particular  $(\varphi|_{\Delta_0}, \{\alpha_x\}_{x \in \Delta_0})$  is a morphism that induces  $\alpha$ .*

*Proof.* Since  $\Delta_0 = \bigcup_{b \in B} \{x \in X : \|\alpha(b)(x)\| > 0\}$  and  $A$  is a continuous  $C_0(X)$ -algebra, we see that set  $\Delta_0$  is open. Suppose that  $x_0$  is a point in the boundary of  $\Delta_0$  in  $\Delta$ . Take a net  $\{x_i\}_i \subseteq \Delta_0$  converging to  $x_0$  and an element  $b \in B$  such that  $\|b(\varphi(x_0))\| = 1$ . Since the homomorphisms  $\alpha_{x_i}$  are isometric, and the mappings  $\Delta \ni x \mapsto \|\alpha_x(b(\varphi(x)))\|$  and  $\Delta \ni x \mapsto \|b(\varphi(x))\|$  are continuous, we get

$$\|\alpha_{x_0}(b(\varphi(x_0)))\| = \lim_i \|\alpha_{x_i}(b(\varphi(x_i)))\| = \lim_i \|b(\varphi(x_i))\| = 1.$$

Hence  $\alpha_{x_0} \neq 0$ , that is  $x_0 \in \Delta_0$ . Thus  $\Delta_0$  is closed in  $\Delta$  and therefore  $\varphi|_{\Delta_0} : \Delta_0 \rightarrow X$  is a proper map. Clearly,  $\alpha$  satisfies (22) with  $\Delta_0$  in place of  $\Delta$ . Accordingly,  $\alpha$  is induced by the morphism  $(\varphi|_{\Delta_0}, \{\alpha_x\}_{x \in \Delta_0})$  by Proposition 3.2.  $\square$

We have the following characterizations of homomorphism induced by morphisms phrased in terms of  $C_0(X)$ -algebras.

**Proposition 3.5.** *Let  $A$  be  $C_0(X)$ -algebra and  $B$  a  $C_0(Y)$ -algebra. For any homomorphism  $\alpha : B \rightarrow A$  the following conditions are equivalent:*

- (i)  $\alpha$  is induced by a morphism from  $\mathcal{B} = \bigsqcup_{y \in Y} B(y)$  to  $\mathcal{A} = \bigsqcup_{x \in X} A(x)$ ,

(ii) there is a homomorphism  $\Phi : C_0(Y) \rightarrow C_0(X)$  such that

$$\alpha(f \cdot b) = \Phi(f) \cdot \alpha(b), \quad f \in C_0(Y), \quad b \in B.$$

If additionally  $B$  is unital and  $A$  is a continuous  $C_0(X)$ -algebra then the above conditions are equivalent to the following one:

(iii)  $\alpha$  maps  $C_0(Y)$  ‘almost into’  $C_0(X)$ , that is

$$\alpha(C_0(Y) \cdot 1) \subseteq C_0(X) \cdot \alpha(1).$$

If the additional assumptions and condition (iii) are satisfied, then the corresponding morphism  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  can be chosen so that  $\Delta$  is compact and each  $\alpha_x$ ,  $x \in \Delta$ , is non-zero.

*Proof.* (i)  $\Rightarrow$  (ii). It suffices to put  $\Phi(a) := a \circ \varphi$  for  $a \in C_0(Y)$ .

(ii)  $\Rightarrow$  (i). Note that  $\Phi : C_0(Y) \rightarrow C_0(X)$  is given by the formula

$$(25) \quad \Phi(b)(x) = \begin{cases} b(\varphi(x)), & x \in \Delta, \\ 0 & x \notin \Delta, \end{cases} \quad b \in C_0(Y)$$

where  $\varphi : \Delta \rightarrow Y$  is a continuous proper mapping defined on an open set  $\Delta \subseteq X$ . Let  $x \in \Delta$ . We define a homomorphism  $\alpha_x : B(\varphi(x)) \rightarrow A(x)$  as follows. For any  $b_0 \in B(\varphi(x))$  there is  $b \in B$  such that  $b(\varphi(x)) = b_0$ , and we claim that the element

$$(26) \quad \alpha_x(b_0) := \alpha(b)(x)$$

is well defined (does not depend on the choice of  $b$ ). Indeed, let  $\tilde{b}, b \in B$  be such that  $\tilde{b}(\varphi(x)) = b(\varphi(x)) = b_0$ . Then  $b(\varphi(x)) - \tilde{b}(\varphi(x)) = 0$ . Upper semicontinuity of the  $C^*$ -bundle  $\mathcal{B} = \bigsqcup_{y \in Y} B(y)$  imply that for every  $\varepsilon > 0$  there is an open neighbourhood  $U$  of  $\varphi(x)$  such that

$$\|b(y) - \tilde{b}(y)\| < \varepsilon, \quad \text{for all } y \in U.$$

Let us choose a function  $h \in C_0(Y)$  such that  $h(\varphi(x)) = 1$ ,  $0 \leq h \leq 1$  and  $h(y) = 0$  outside  $U$ . We get

$$\begin{aligned} \|\alpha(b)(x) - \alpha(\tilde{b})(x)\| &= \|(\Phi(h)\alpha(b) - \Phi(h)\alpha(\tilde{b}))(x)\| = \|\alpha(hb - h\tilde{b})(x)\| \\ &\leq \|\alpha(hb - h\tilde{b})\| \leq \|hb - h\tilde{b}\| \leq \varepsilon. \end{aligned}$$

This proves our claim. Now it is straightforward to see that (26) gives the desired homomorphism  $\alpha_x : B(\varphi(x)) \rightarrow A(x)$ . Moreover, for the above defined pair  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  the formula (22) holds. Hence in view of Proposition 3.2,  $\alpha$  is induced by a morphism.

Let us now assume that  $B$  is a unital and  $A$  is a continuous  $C_0(X)$ -algebra.

(ii)  $\Rightarrow$  (iii). It is obvious.

(iii)  $\Rightarrow$  (ii). Since  $\alpha(C_0(Y) \cdot 1) \subseteq C_0(X) \cdot \alpha(1)$ , for every  $f \in C_0(Y)$  there exists  $g \in C_0(X)$  such that

$$\alpha(f \cdot 1)(x) = g(x)\alpha(1)(x), \quad x \in X.$$

Clearly, the function  $g$  is uniquely determined by  $f$  on the set  $\Delta := \{x \in X : \alpha(B)(x) \neq 0\} = \{x \in X : \alpha(1)(x) \neq 0\}$ . Since the mapping  $X \ni x \rightarrow \|\alpha(1)(x)\| \in \{0, 1\}$  is continuous and vanishing at infinity,  $\Delta$  is open and compact. Now it is straightforward to see that the formula  $\Phi(f) = g|_\Delta$  defines a homomorphism  $\Phi : C_0(Y) \rightarrow C(\Delta) \subseteq C_0(X)$  satisfying condition (ii).  $\square$

**Example 3.6.** Suppose that  $q_I : A \rightarrow A/I$  is a quotient map and  $A$  is a  $C_0(X)$ -algebra. We may treat  $A/I$  as a  $C_0(V)$ -algebra for any closed set  $V$  containing  $\sigma_A(\text{Prim}(A/I))$ , cf. Lemma 1.7. Then we have

$$q_I(f \cdot a) = f|_V \cdot q_I(a), \quad f \in C_0(X), \quad a \in A.$$

Hence condition (ii) in Proposition 3.5 is satisfied. In particular,  $q_I$  is induced by the morphism  $(id, \{q_{I,x}\}_{x \in V})$  where  $q_{I,x} : A(x) \rightarrow A(x)/I(x)$ ,  $x \in V$ , are the quotient maps.

Let us consider a category of  $C_0(X)$ -algebras with morphisms being homomorphisms satisfying the equivalent conditions in Proposition 3.5. In this paper, we are interested in properties of systems  $(A, \alpha)$  where  $A$  is an object and  $\alpha$  is a morphism in this category.

**Definition 3.7.** We say that a  $C^*$ -dynamical system  $(A, \alpha)$  is a  $C_0(X)$ -dynamical system, if  $A$  is a  $C_0(X)$ -algebra and  $\alpha$  is induced by a morphism. If additionally  $A$  is a continuous  $C_0(X)$ -algebra, we say that  $(A, \alpha)$  is a *continuous  $C_0(X)$ -dynamical system*

In section 5, we will study crossed products associated to continuous  $C_0(X)$ -dynamical systems introduced in the following example.

**Example 3.8** (Endomorphisms of  $C^*$ -algebras with Hausdorff primitive ideal space). If  $A$  is a  $C^*$ -algebra and its primitive ideal space  $X := \text{Prim}(A)$  is Hausdorff, then using Dauns-Hofmann isomorphism we may naturally treat  $A$  as a continuous  $C_0(X)$ -algebra where the structure map  $\sigma_A$  is identity, cf. [6, 2.2.2]. In particular, for  $x \in X = \text{Prim}(A)$  the fiber  $A(x) = A/x$  is a simple (non-zero)  $C^*$ -algebra. Thus if  $\alpha : A \rightarrow A$  is an endomorphism induced by a morphism  $(\varphi, \{\alpha_x\}_{x \in \Delta})$ , then Lemma 3.4 applies and we may assume that each  $\alpha_x$ ,  $x \in \Delta$ , is injective. Moreover, if  $A$  is unital then we may identify  $C(X)$  with  $Z(A)$  and by Proposition 3.5 an endomorphism  $\alpha : A \rightarrow A$  is induced by a morphism if and only if  $\alpha(Z(A)) \subseteq Z(A)\alpha(1)$ .

For trivial  $C^*$ -bundles we have the following description of endomorphisms induced by morphisms. We equip the set  $\text{End}(D)$  of all endomorphisms of a  $C^*$ -algebra  $D$  with the topology of point-wise convergence.

**Proposition 3.9.** *Let  $\varphi : \Delta \rightarrow X$  be a proper continuous mapping defined on an open set  $\Delta \subseteq X$  and let a continuous mapping  $\Delta \ni x \rightarrow \alpha_x \in \text{End}(D)$  where  $D$  is a  $C^*$ -algebra. We treat  $A := C_0(X, D)$  as a  $C_0(X)$ -algebra in an obvious way. The formula*

$$(27) \quad \alpha(a)(x) := \begin{cases} \alpha_x(a(\varphi(x))), & x \in \Delta, \\ 0 & x \notin \Delta, \end{cases} \quad a \in C_0(X, D), \quad x \in X.$$

*defines an endomorphism of  $A$  induced by a morphism. Every endomorphism of  $A$  induced by a morphism arises in this way. If  $D$  is simple, or if  $A$  is unital, then we can choose the corresponding morphism in such a way that each  $\alpha_x$ ,  $x \in \Delta$ , is non-zero.*

*Proof.* The corresponding  $C^*$ -bundle  $\mathcal{A} = \bigsqcup_{x \in X} D$  can be identified with the product  $X \times D$ , together with its product topology. In this case condition 2b) from Definition 3.1 translates to the following: If  $\{x_i\}_{i \in \Lambda} \subseteq \Delta$  and  $\{b_i\}_{i \in \Lambda} \subseteq D$  are nets such that  $x_i \rightarrow x \in \Delta$  and

$b_i \rightarrow b \in D$ , then  $\alpha_{x_i}(b_i) \rightarrow \alpha_x(b)$ . The latter condition is equivalent to the continuity of the map  $\Delta \ni x \rightarrow \alpha_x \in \text{End}(D)$ , which can be readily deduced from the inequality:

$$\|\alpha_{x_i}(b_i) - \alpha_x(b)\| \leq \|b_i - b\| + \|\alpha_{x_i}(b) - \alpha_x(b)\|.$$

Thus the assertion follows by Proposition 3.2. The last remark follows by Lemma 3.4 and the last part of Proposition 3.2.  $\square$

**3.2. Quotients and restrictions of  $C_0(X)$ -dynamical systems.** Restrictions and quotients of  $C_0(X)$ -dynamical systems can be treated as  $C_0(X)$ -dynamical systems in the following sense.

**Proposition 3.10.** *Suppose that  $\alpha : A \rightarrow A$  is an endomorphism induced by a morphism  $(\varphi, \{\alpha_x\}_{x \in \Delta})$ . Let  $I$  be a positively invariant ideal in  $(A, \alpha)$ . Then  $(I, \alpha|_I)$  and  $(A/I, \alpha_I)$  are naturally  $C_0(X)$ -dynamical systems where  $\alpha|_I$  is induced by  $(\varphi, \{\alpha_x|_{I(\varphi(x))}\}_{x \in \Delta})$  and  $\alpha_I$  is induced by  $(\varphi, \{\alpha_{I,x}\}_{x \in \Delta})$  where*

$$(28) \quad \alpha_{I,x}(a + I(\varphi(x))) := \alpha_x(a) + I(x), \quad a \in A(\varphi(x)), \quad x \in \Delta.$$

*Proof.* Note that positive invariance of  $I$  implies that  $\alpha_x(I(\varphi(x))) \subseteq I(x)$ ,  $x \in \Delta$ . In particular, (28) gives a well defined homomorphism  $\alpha_{I,x}$ . Now, Proposition 3.2 readily implies that  $(\varphi, \{\alpha_x|_{I(\varphi(x))}\}_{x \in \Delta})$  is a morphism that induces  $\alpha|_I$  and that  $(\varphi, \{\alpha_{I,x}\}_{x \in \Delta})$  is a morphism that induces  $\alpha^I$  (cf. the description of the quotient  $C^*$ -bundle in Lemma 1.7).  $\square$

We note that even when the structure map  $\mu_A : C_0(X) \rightarrow Z(M(A))$  is injective, the structure maps for  $I$  and  $A/I$  treated as  $C_0(X)$ -algebras as in the above proposition, will hardly ever be injective. Moreover,  $A/I$  might not be a continuous  $C_0(X)$ -algebra, even if  $A$  is, cf. Lemma 1.7. In certain situations, these problems can be circumvented by using the following proposition.

**Proposition 3.11.** *Suppose that  $\alpha : A \rightarrow A$  is an endomorphism induced by a morphism  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  and  $I$  is positively invariant ideal in  $(A, \alpha)$ . Put*

$$V := \overline{\sigma_A(\text{Prim}(A/I))}, \quad U := \sigma_A(\text{Prim}(I))$$

*and treat  $A/I$  as a  $C_0(V)$ -algebra and  $I$  as a  $C_0(U)$ -algebra.*

- (i) *If  $\varphi(V \cap \Delta) \subseteq V$ , which is automatic when for each  $x \in \Delta$  the range of  $\alpha_x$  is a full subalgebra of  $A(x)$ , then the quotient endomorphism  $\alpha_I : A/I \rightarrow A/I$  is induced by the morphism  $(\varphi|_{V \cap \Delta}, \{\alpha_{I,x}\}_{x \in V \cap \Delta})$ , cf. (28).*
- (ii) *If  $U$  is open and  $\varphi^{-1}(U) \subseteq U$ , which is automatic when  $A$  is a continuous  $C_0(X)$ -algebra and each  $\alpha_x$ ,  $x \in \Delta$ , is injective, then the restricted  $C^*$ -dynamical system  $(I, \alpha|_I)$  is a naturally induced by a morphism  $(\varphi|_{\varphi^{-1}(U)}, \{\alpha_x|_{I(\varphi(x))}\}_{x \in \varphi^{-1}(U)})$ .*

*Proof.* (i). Suppose that  $\varphi(V \cap \Delta) \subseteq V$ . Then the restriction  $\varphi|_V : V \cap \Delta \rightarrow V$  is a proper map and  $A$  can be naturally treated as a  $C_0(V)$ -algebra by Lemma 1.7. Hence the morphism  $(\varphi, \{\alpha_{I,x}\}_{x \in \Delta})$  from Proposition 3.10 restricts to a morphism  $(\varphi|_{V \cap \Delta}, \{\alpha_{I,x}\}_{x \in V \cap \Delta})$  that induces  $\alpha_I$ .

Now, we show that  $\varphi(V \cap \Delta) \subseteq V$ , if for each  $x \in \Delta$  the range of  $\alpha_x$  is a full subalgebra of  $A(x)$ . To this end, let  $V_0 = \sigma_A(\text{Prim}(A/I))$  and recall that  $x \in V_0$  if and only if  $I(x) \neq A(x)$ , see (3). Let  $x \in \Delta \cap V_0$ . We claim that  $\varphi(x) \in V_0$ . Indeed, assume on the contrary that

$I(\varphi(x)) = A(\varphi(x))$ . Then by positive invariance of  $I$  we have  $\alpha_x(A(\varphi(x))) = \alpha_x(I(\varphi(x))) \subseteq I(x)$ . Since  $\alpha_x(A(\varphi(x)))$  is full in  $A(x)$  we get

$$I(x) = A(x)I(x)A(x) \supseteq A(x)\alpha_x(A(\varphi(x)))A(x) = A(x).$$

This contradicts the fact that  $x \in Y_0$ , cf. (3). Accordingly,  $\varphi(V_0 \cap \Delta) \subseteq V_0$ . By continuity of  $\varphi$  we get  $\varphi(V \cap \Delta) \subseteq V$ .

(ii). If  $U$  is open and  $\varphi^{-1}(U) \subseteq U$ , then  $\varphi|_{\varphi^{-1}(U)} : \varphi^{-1}(U) \rightarrow U$  is a proper map and  $I$  is naturally a  $C_0(U)$ -algebra. Since  $U = \{x \in X : I(x) \neq \{0\}\}$  by (2), for any  $a \in I$  and  $x \notin \varphi^{-1}(U)$  we have  $\alpha(a)(x) = 0$ . Thus  $(\varphi|_{\varphi^{-1}(U)}, \{\alpha_x|_{I(\varphi(x))}\}_{x \in \varphi^{-1}(U)})$  is a morphism that induces  $(I, \alpha|_I)$ , by Proposition 3.5.

Now, suppose that  $A$  is a continuous  $C_0(X)$ -algebra and each  $\alpha_x$ ,  $x \in \Delta$ , is injective. Then  $U$  is open. By (2), for each  $x \in \varphi^{-1}(U)$  we may find an element  $a \in I$  such that  $a(\varphi(x)) \neq 0$ . Since  $\alpha_x$  is injective, we have  $\alpha(a)(x) = \alpha_x(a(\varphi(x))) \neq 0$ , and thus  $x \in U$ , again by (2). Hence  $\varphi^{-1}(U) \subseteq U$ .  $\square$

The above proposition implies that the quotient and the restriction of continuous  $C_0(X)$ -dynamical systems described in Example 3.8 are naturally  $C^*$ -dynamical systems of the same type, see Lemma 5.3 below.

**3.3. Extendible morphisms and reversible  $C_0(X)$ -dynamical systems.** In the foregoing lemma we use the description of multiplier algebras given in Proposition 1.6.

**Lemma 3.12.** *Suppose that  $A$  is a  $C_0(X)$ -algebra,  $B$  is a  $C_0(Y)$ -algebra, and  $\alpha : B \rightarrow A$  is an extendible homomorphism induced by a morphism  $(\varphi, \{\alpha_x\}_{x \in \Delta})$ . Each  $\alpha_x : B(\varphi(x)) \rightarrow A(x)$ ,  $x \in \Delta$ , is extendible and  $\overline{\alpha} : M(B) \rightarrow M(A)$  is given by*

$$(29) \quad \overline{\alpha}(m)(x) = \begin{cases} \overline{\alpha}_x(m(\varphi(x))), & x \in \Delta, \\ 0_x, & x \notin \Delta, \end{cases} \quad m \in M(B), x \in X.$$

*If additionally either  $A$  has local units or  $A$  is locally trivial then the set  $\Delta_0$  defined in (24) is closed in  $X$ .*

*Proof.* Let  $\{\mu_\lambda\}$  be an approximate unit in  $B$  and let  $x \in \Delta$ . Then  $\{\mu_\lambda(\varphi(x))\}$  is an approximate unit in  $B(\varphi(x))$  and  $\alpha_x(\mu_\lambda(\varphi(x))) = \alpha(\mu_\lambda)(x)$  converges strictly in  $A(x)$ . Hence the homomorphisms  $\alpha_x$ ,  $x \in \Delta$ , are extendible. Recall that  $\overline{\alpha}$  is determined by the formula  $\overline{\alpha}(m)a = \lim_\lambda \alpha(m\mu_\lambda)a$  where  $a \in A$ ,  $m \in M(B)$ . It follows that for any  $x \in \Delta$  we have

$$\begin{aligned} (\overline{\alpha}(m)a)(x) &= \lim_\lambda \alpha(m\mu_\lambda)(x)a(x) = \lim_\lambda \alpha_x((m\mu_\lambda)(\varphi(x)))a(x) \\ &= \lim_\lambda \alpha_x(m(\varphi(x))\mu_\lambda(\varphi(x)))a(x) = \overline{\alpha}_x(m(\varphi(x)))a(x). \end{aligned}$$

Thus we get (29).

Now suppose that  $x_0$  is a point in the boundary of the set  $\Delta_0 = \{x \in X : \alpha(B)(x) \neq 0\}$ , but  $x_0 \notin \Delta_0$ . If  $A$  has local units, or if  $A$  is locally trivial, then we may choose  $a \in A$ , such that  $\|a(x)\| = 1$  for every  $x$  in an open neighbourhood  $U$  of  $x_0$ . Then the compact set  $\{x \in X : \|(\overline{\alpha}(1)a)(x)\| \geq 1/2\}$  contains  $\Delta_0 \cap U$  but does not contain  $x_0$ . This leads to a contradiction, since  $x_0$  is in the closure of  $\Delta_0 \cap U$ .  $\square$

Suppose now that  $(A, \alpha)$  is a reversible  $C^*$ -dynamical system where  $A$  is a  $C_0(X)$ -algebra and  $\alpha$  is induced by a morphism  $(\varphi, \{\alpha_x\}_{x \in \Delta})$ . In general all we can say about the kernel of  $\alpha$  and its annihilator is that

$$\ker \alpha = \{a \in A : a(y) \in \bigcap_{x \in \varphi^{-1}(y)} \ker \alpha_x \text{ for all } y \in \varphi(\Delta)\}$$

and  $(\ker \alpha)^\perp$  is contained in

$$(30) \quad \{a \in A : a|_{X \setminus \varphi(\Delta)} = 0 \text{ and } a(y) \in \left( \bigcap_{x \in \varphi^{-1}(y)} \ker \alpha_x \right)^\perp \text{ for } y \in \varphi(\Delta)\}.$$

Nevertheless, we have the following statement.

**Proposition 3.13.** *Suppose that  $(A, \alpha)$  is a reversible  $C^*$ -dynamical system where  $A$  is a  $C_0(X)$ -algebra and  $\alpha$  is induced by a morphism  $(\varphi, \{\alpha_x\}_{x \in \Delta})$ .*

- (i) *If all of the endomorphisms  $\alpha_x$ ,  $x \in \Delta$ , are injective then  $\varphi$  is injective.*
- (ii) *If  $\varphi$  is injective and  $(A, \alpha)$  is extendible then the unique regular transfer operator for  $(A, \alpha)$  is determined by the formula*

$$(31) \quad \alpha_*(a)(x) = \begin{cases} \alpha_{*,x}(a(\varphi^{-1}(x))), & x \in \varphi(\Delta), \\ 0_x, & x \notin \varphi(\Delta), \end{cases} \quad a \in A, \ x \in X,$$

where for each  $x \in \varphi(\Delta)$ ,  $\alpha_{*,x} : A(\varphi^{-1}(x)) \rightarrow A(x)$  is a completely positive generalized inverse to  $\alpha_{\varphi^{-1}(x)} : A(x) \rightarrow A(\varphi^{-1}(x))$ . The mappings  $\alpha_{*,x}$  have strictly continuous extensions  $\overline{\alpha}_{*,x}$  and the strictly continuous extension  $\overline{\alpha}_*$  of  $\alpha_*$  is given by the formula

$$\overline{\alpha}_*(a)(x) = \begin{cases} \overline{\alpha}_{*,x}(a(\varphi^{-1}(x))), & x \in \varphi(\Delta), \\ 0_x, & x \notin \varphi(\Delta), \end{cases} \quad a \in M(A), \ x \in X,$$

where we use the description of multipliers given in Proposition 1.6.

*Proof.* (i). Injectivity of  $\alpha_x$ 's imply that  $\ker \alpha = \{a \in A : a(x) = 0 \text{ for all } x \in \varphi(\Delta)\}$  and therefore  $(\ker \alpha)^\perp \subseteq \{a \in A : a(x) = 0 \text{ for all } x \notin \varphi(\Delta)\}$ . Let  $x, y \in \Delta$  be two different points. Take any  $b \in \alpha(A)$  such that  $b(x) \neq 0$  and any  $h \in C_0(X)$  such that  $h(x) = 1$  and  $h(y) = 0$ . Since  $\alpha(A) = \alpha(A)A\alpha(A)$  we see that  $c := hb$  is in  $\alpha(A)$ . Obviously,  $c(x) \neq 0$  and  $c(y) = 0$ . Thus for any  $a \in A$  with  $\alpha(a) = c$  we have  $\alpha_x(a(\varphi(x))) \neq 0$  and  $\alpha_y(a(\varphi(y))) = 0$ . Injectivity of  $\alpha_x$  and  $\alpha_y$  implies that  $\varphi(x) \neq \varphi(y)$ . Hence  $\varphi$  is injective.

(ii). Fix  $x \in \Delta$  and  $b_0 \in \overline{\alpha}_x(1_{\varphi(x)})A(x)\overline{\alpha}_x(1_{\varphi(x)})$  (we use the notation of Lemma 3.12). Take  $b \in \alpha(A) = \overline{\alpha}(1)A\overline{\alpha}(1)$  such that  $b(x) = b_0$ . Let  $a \in (\ker \alpha)^\perp$  be the unique element such that  $\alpha(a) = b$ . Accordingly,  $b_0 = b(x) = \alpha_x(a(\varphi(x)))$  where  $a(\varphi(x)) \in (\ker \alpha_x)^\perp$  (here we use injectivity of  $\varphi$  and that  $(\ker \alpha)^\perp$  is contained in the set (30)). It follows that the range of  $\alpha_x : A(\varphi(x)) \rightarrow A(x)$  is the corner  $\overline{\alpha}_x(1_{\varphi(x)})A(x)\overline{\alpha}_x(1_{\varphi(x)})$  and  $\alpha_x : (\ker \alpha_x)^\perp \rightarrow \alpha_x(A(\varphi(x)))$  is an isomorphism. The latter fact implies that  $\ker \alpha_x$  is a complemented ideal in  $A(\varphi(x))$ . We define the map

$$(32) \quad \alpha_{*,\varphi(x)}(a) := \alpha_x^{-1}\left(\overline{\alpha}_x(1_{\varphi(x)})a\overline{\alpha}_x(1_{\varphi(x)})\right), \quad a \in A(x),$$



where  $\alpha_x^{-1}$  is the inverse to the isomorphism  $\alpha_x : (\ker \alpha_x)^\perp \rightarrow \overline{\alpha}_x(1_{\varphi(x)})A(x)\overline{\alpha}_x(1_{\varphi(x)})$ . The maps  $\alpha_{*,x}$  have strictly continuous extensions which are given by (32) with  $\alpha_x^{-1}$  replaced by the inverse to the strictly continuous isomorphism  $\overline{\alpha}_x : M((\ker \alpha_x)^\perp) \rightarrow \overline{\alpha}_x(1_{\varphi(x)})M(A(x))\overline{\alpha}_x(1_{\varphi(x)})$ , cf. Lemma 3.12. Now it is immediate to see that the homomorphisms  $\alpha_{*,x}$  and  $\overline{\alpha}_{*,x}$ ,  $x \in \varphi(\Delta)$ , fulfill the requirements of the assertion.  $\square$

Injectivity of the map  $\varphi$  in the second part of the above proposition is essential.

**Example 3.14.** Consider a reversible  $C^*$ -dynamical system  $(A, \alpha)$  where  $A = \mathbb{C}^3$  and  $\alpha(a) = (a_2, 0, a_3)$  for  $a = (a_1, a_2, a_3) \in A$ . Then the regular transfer operator  $\alpha_*(a) = (0, a_1, a_3)$  for  $(A, \alpha)$  is actually an endomorphism. Treating  $A$  as a  $C(\{1, 2\})$ -algebra where  $a(1) = a_1 \in \mathbb{C}$  and  $a(2) = (a_2, a_3) \in \mathbb{C}^2$ , for  $a \in A$ , the endomorphism  $\alpha$  is induced by the morphism  $(\varphi, \{\alpha_1, \alpha_2\})$  where

$$\varphi(1) = \varphi(2) = 2, \quad \alpha_1(a_2, a_3) = a_2, \quad \alpha_2(a_2, a_3) = (0, a_3).$$

But  $\alpha_*$  is not induced by a morphism because the fiber  $\alpha_*(a)(2) = (a_1, a_3)$  of  $\alpha_*(a)$  depends on two fibers of  $a$ .

**3.4. Direct limits.** In this subsection, we show that a direct limit of  $C_0(X_n)$ -algebras is naturally a  $C_0(\tilde{X})$ -algebra, and in certain cases a continuous  $C_0(\tilde{X})$ -algebra. We will use this result, in subsection 4.2, to show that a reversible extension of a  $C_0(X)$ -dynamical system is a  $C_0(\tilde{X})$ -dynamical system.

Let us consider a direct sequence  $B_0 \xrightarrow{\alpha_0} B_1 \xrightarrow{\alpha_1} \dots$ , where for each  $n \in \mathbb{N}$ ,  $B_n$  is a  $C_0(X_n)$ -algebra and  $\alpha_n : B_n \rightarrow B_{n+1}$  is a homomorphism induced by a morphism  $(\varphi_n, \{\alpha_x\}_{x \in X_{n+1}})$  (we may assume that the sets  $X_n$  are disjoint, so a homomorphism  $\alpha_x$  is uniquely determined by the point  $x \in X_{n+1}$ ). We show that the  $C^*$ -algebraic direct limit  $B := \varinjlim \{B_n, \alpha_n\}$  is naturally a  $C_0(\tilde{X})$ -algebra over the topological inverse limit space

$$\tilde{X} := \varprojlim \{X_{n+1}, \varphi_n\}.$$

To this end, we attach to each  $\tilde{x} = (x_0, x_1, \dots) \in \tilde{X}$  the direct limit  $C^*$ -algebra

$$B(\tilde{x}) := \varinjlim \{B_n(x_n), \alpha_{x_{n+1}}\}$$

of the direct sequence  $B_0(x_0) \xrightarrow{\alpha_{x_1}} B_1(x_1) \xrightarrow{\alpha_{x_2}} B_2(x_2) \xrightarrow{\alpha_{x_3}} \dots$ . We let  $\phi_{\tilde{x},n} : B_n(x_n) \rightarrow B(\tilde{x})$  and  $\phi_n : B_n \rightarrow B$  be the natural homomorphisms:

$$\phi_n(b_n) = [0, \underbrace{\dots, 0}_n, b_n, \alpha_{n+1}(b_n), \dots],$$

$$\phi_{\tilde{x},n}(b_n) = [0, \underbrace{\dots, 0}_n, b_n(x_n), \alpha_{x_{n+1}}(b_n(x_n)), \dots]$$

where  $b_n \in B_n$  and  $\tilde{x} \in \tilde{X}$ . The following statement can be viewed as a generalization of [18, Proposition 1.7] to the non-unital case. In contrast to [18] we prove it using the  $C^*$ -bundle approach.

**Proposition 3.15.** *Retain the above notation and suppose additionally that the mappings  $\varphi_n$  are surjective,  $n \in \mathbb{N}$ . There is a unique topology on  $\mathcal{B} = \bigsqcup_{\tilde{x} \in \tilde{X}} B(\tilde{x})$  making it an upper semicontinuous  $C^*$ -bundle over  $\tilde{X}$  such that*

$$(33) \quad B \ni \phi_n(b_n) \longrightarrow \phi_n(b_n)(\tilde{x}) := \phi_{\tilde{x},n}(b_n(x_n)), \quad \tilde{x} \in \tilde{X}, b_n \in B_n,$$

*establishes the natural isomorphism from  $B = \varinjlim \{B_n, \alpha_n\}$  onto  $\Gamma_0(\mathcal{B})$ .*

*If additionally, all the algebras  $B_n$  are continuous  $C_0(X_n)$ -algebras and all the endomorphisms  $\alpha_x$ ,  $x \in X_{n+1}$ ,  $n \in \mathbb{N}$ , are injective, then the  $C^*$ -bundle  $\mathcal{B} = \bigsqcup_{\tilde{x} \in \tilde{X}} B(\tilde{x})$  is continuous.*

*Proof.* For  $\tilde{x} \in \tilde{X}$  and  $m > n$  we put

$$\alpha_{\tilde{x},[n,m]} := \alpha_{x_m} \circ \dots \circ \alpha_{x_{n+2}} \circ \alpha_{x_{n+1}} \quad \text{and} \quad \alpha_{[n,m]} := \alpha_{m-1} \circ \dots \circ \alpha_{n+1} \circ \alpha_n.$$

These are the bonding homomorphisms from  $B(x_n)$  to  $B(x_m)$  and from  $B_n$  to  $B_m$ , respectively. Let  $\tilde{x} \in \tilde{X}$ ,  $b_n \in B_n$ . To check that the map (33) is well defined assume that  $\phi_n(b_n) = 0$ . Then for any  $\varepsilon > 0$  and sufficiently large  $m$  we have  $\|\alpha_{[n,m]}(b_n)\| < \varepsilon$ , and all the more  $\|\alpha_{\tilde{x},[n,m]}(b_n(x_n))\| = \|\alpha_{[n,m]}(b_n)(x_m)\| < \varepsilon$ . This implies that  $\phi_{\tilde{x},n}(b_n(x_n)) = 0$ . Hence (33) is well defined and clearly it yields a surjective homomorphism from  $B$  onto  $B(\tilde{x})$ .

We show that for a fixed  $\phi_n(b_n) \in B$ , the mapping

$$(34) \quad \tilde{X} \ni \tilde{x} \mapsto \|\phi_n(b_n)(\tilde{x})\| \in \mathbb{C}$$

is upper semicontinuous. Suppose that  $\tilde{x} \in \tilde{X}$  is such that  $\|\phi_n(b_n)(\tilde{x})\| < K$ . Then there is  $m > n$  such that  $\|\alpha_{\tilde{x},[n,m]}(b_n(x_n))\| < K$ . Since  $\alpha_{[m,n]}(b_n)(x_m) = \alpha_{\tilde{x},[n,m]}(b_n(x_n))$  and  $X_m \ni x \rightarrow \|\alpha_{[m,n]}(b_n)(x)\|$  is upper semicontinuous, there is an open neighborhood  $U$  of  $x_m$  such that  $\|\alpha_{[m,n]}(b_n)(x)\| < K$  for all  $x \in U$ . It follows that the set

$$\tilde{U} := \{\tilde{y} = (y_0, y_1, \dots) \in \tilde{X} : y_m \in U\}$$

is an open neighborhood of  $\tilde{x}$  such that for  $\tilde{y} \in \tilde{U}$  we have

$$\|\phi_n(b_n)(y)\| \leq \|\alpha_{\tilde{y},[n,m]}(b_n(y_n))\| < K.$$

This proves the upper semicontinuity of (34).

We wish to show that (34) vanishes at infinity. Let  $\varepsilon > 0$ . By upper semicontinuity of (34) the set  $\{\tilde{x} \in \tilde{X} : \|\phi_n(b_n)(\tilde{x})\| \geq \varepsilon\}$  is closed, and clearly, it is a subset of  $\{\tilde{x} \in \tilde{X} : \|b_n(x_n)\| \geq \varepsilon\}$ . However, the latter set is compact because the map  $\tilde{X} \ni \tilde{x} \rightarrow x_n \in X_n$  is proper and  $X_n \ni x \rightarrow \|b_n(x)\|$  is vanishing at the infinity. Hence  $\{\tilde{x} \in \tilde{X} : \|\phi_n(b_n)(\tilde{x})\| \geq \varepsilon\}$  is compact as well.

Now, by Fell's theorem, see [55, Theorem C.25], there is a unique topology on  $\mathcal{B}$  such that (33) defines a surjective homomorphism from  $B$  onto  $\Gamma_0(\mathcal{B})$ . We still need to show that this homomorphism is injective.

To this end, assume that  $\phi_n(b_n)$  is non zero. Then there exists  $\varepsilon > 0$  such that  $\|\alpha_{[m,n]}(b_n)\| > \varepsilon$  for all  $m > n$ . Thus, for each  $m > n$ , the set

$$D_m := \{x \in X_m : \|\alpha_{[m,n]}(b_n)(x)\| \geq \varepsilon\}$$

is nonempty, and it is compact because  $X_m \ni x \rightarrow \|\alpha_{[m,n]}(b_n)(x)\|$  vanishes at infinity. Note that  $\varphi_m(D_{m+1}) \subseteq D_m$ . Thus the sets

$$\tilde{D}_m := \{\tilde{x} \in \tilde{X} : x_m \in D_m\}$$

form a decreasing sequence of compact nonempty sets (non-emptiness follows from surjectivity of the mappings  $\varphi_m$ ). Hence there is  $\tilde{x}_0 \in \bigcap_{m>n} \tilde{D}_m$  and plainly

$$\|\phi_n(b_n)(\tilde{x}_0)\| \geq \varepsilon > 0.$$

This finishes the proof of the first part of the assertion.

Assume now that for each  $n \in \mathbb{N}$ ,  $B_n$  is a continuous  $C_0(X_n)$ -algebra and all of the endomorphisms  $\alpha_x$ ,  $x \in X_n$ , are injective. Then  $\|\phi_n(b_n)(\tilde{x})\| = \|b_n(x_n)\|$  for all  $\tilde{x} \in \tilde{X}$ ,  $b_n \in B_n$ ,  $n \in \mathbb{N}$ . Hence mapping (34), as a composition of two continuous mappings  $\tilde{X} \ni \tilde{x} \rightarrow x_n \in X_n$  and  $X_n \ni x_n \rightarrow \|b_n(x_n)\|$ , is continuous.  $\square$

Injectivity of the endomorphisms  $\alpha_x$ ,  $x \in X_{n+1}$ , in the second part of Proposition 3.15 is essential.

**Example 3.16.** Consider the stationary inductive limit given by the continuous  $C_0(\mathbb{N})$ -algebra  $A := C_0(\mathbb{N}, \mathbb{C}^2)$  and the endomorphism  $\alpha : A \rightarrow A$  induced by the morphism  $(\varphi, \{\alpha_n\}_{n \in \mathbb{N}})$  where

$$\phi(0) = 0, \quad \alpha_0 = id, \quad \text{and} \quad \phi(n) = n - 1, \quad \alpha_n(a, b) = (a, 0), \quad \text{for } n > 0.$$

The resulting direct limit  $B = \varinjlim \{A, \alpha\}$  can be viewed as a  $C_0(\{-\infty\} \cup \mathbb{Z})$ -algebra with the obvious topology on  $\{-\infty\} \cup \mathbb{Z}$  and fibers  $B_{-\infty} = \mathbb{C}^2$  and  $B_n = \mathbb{C}$ ,  $n \in \mathbb{Z}$ . The image of the constant function  $\mathbb{N} \ni n \rightarrow (a, b) \in \mathbb{C}^2$  (treated as an element of  $A$ ) in the algebra  $B$  corresponds to the section  $f$  with  $f(-\infty) = (a, b)$  and  $f(n) = a$  for  $n \in \mathbb{Z}$ . If  $|a| < |b|$ , the function  $\{-\infty\} \cup \mathbb{Z} \ni x \rightarrow \|f(x)\|$  is not lower semicontinuous at  $-\infty$ .

#### 4. CROSSED PRODUCTS OF $C_0(X)$ -DYNAMICAL SYSTEMS

In this section, we fix a  $C_0(X)$ -dynamical system  $(A, \alpha)$  and denote by  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  a morphism that induces  $\alpha$ . We also fix an ideal  $J$  in  $(\ker \alpha)^\perp$  and make the following standing assumption:

- $\sigma_A : \text{Prim}(A) \rightarrow X$  is surjective, equivalently  $A(x) \neq \{0\}$  for all  $x \in X$ .

The above assumption allows us to treat  $C_0(X)$  as a subalgebra of  $M(Z(A))$ . We will study crossed products  $C^*(A, \alpha; J)$  using the following tactic. Firstly, we consider reversible systems. Then we show that the natural reversible  $J$ -extension  $(B, \beta)$  of  $(A, \alpha)$  is induced by a morphism, which will immediately lead us to general results.

**4.1. The case of a reversible system.** In this subsection, we assume that  $(A, \alpha)$  is a reversible  $C^*$ -dynamical system. The dual partial homeomorphism  $\hat{\alpha} : \widehat{\alpha(A)} \rightarrow (\ker \alpha)^\perp$ , cf. Definition 2.34, factors through to the partial homeomorphism of the primitive ideal space  $\text{Prim}(A)$ . We denote the latter mapping by  $\check{\alpha} : \text{Prim}(\alpha(A)) \rightarrow \text{Prim}((\ker \alpha)^\perp)$ . Thus we have  $\check{\alpha}(\ker \pi) = \ker(\pi \circ \alpha)$  for  $\pi \in \widehat{\alpha(A)}$ . With the identifications  $\text{Prim}(\alpha(A)) = \{P \in \text{Prim}(A) : \alpha(A) \not\subseteq P\}$  and  $\text{Prim}((\ker \alpha)^\perp) = \{P \in \text{Prim}(A) : (\ker \alpha)^\perp \not\subseteq P\}$  we have

$$(35) \quad \check{\alpha}(P) = \alpha^{-1}(P), \quad P \in \text{Prim}(\alpha(A)).$$

**Lemma 4.1.** *With the above notation, the following diagram*

$$\begin{array}{ccc} \text{Prim}(\alpha(A)) & \xrightarrow{\check{\alpha}} & \text{Prim}((\ker \alpha)^\perp) \\ \sigma_A \downarrow & & \downarrow \sigma_A \\ \Delta & \xrightarrow{\varphi} & \varphi(\Delta) \end{array}$$

*commutes. In particular, if  $\varphi$  is free then  $\check{\alpha}$  is free, and if  $A$  is a continuous  $C_0(X)$ -algebra and  $\varphi$  is topologically free, then  $\check{\alpha}$  is also topologically free.*

*Proof.* Using, among other things, (2) and (30) we see that

$$\begin{aligned} \sigma_A(\text{Prim}(\alpha(A))) &= \sigma_A(\text{Prim}(A\alpha(A)A)) = \{x \in X : (A\alpha(A)A)(x) \neq 0\} \\ &= \{x \in X : \alpha(A)(x) \neq 0\} \subseteq \Delta, \\ \sigma_A(\text{Prim}((\ker \alpha)^\perp)) &= \{x \in (\ker \alpha)^\perp : (x) \neq 0\} \subseteq \varphi(\Delta). \end{aligned}$$

Now let  $P \in \text{Prim}(\alpha(A))$ . Then  $x := \sigma_A(P)$  is in  $\Delta$ . By (1),  $C_0(\Delta \setminus \{x\}) \cdot A \subseteq P$ . Applying to this inclusion  $\alpha^{-1}$  we get  $C_0(\varphi(\Delta) \setminus \{\varphi(x)\}) \cdot \alpha^{-1}(A) \subseteq \alpha^{-1}(P)$ . This in view of (35) and (1) means that  $\sigma_A(\check{\alpha}(P)) = \varphi(x) = \sigma_A(P)$ . The last part of the assertion is straightforward.  $\square$

Clearly, (topological) freeness of  $\check{\alpha}$  implies (topological) freeness of  $\hat{\alpha}$ . Hence Lemma 4.1 and Proposition 2.35 give us the following results.

**Corollary 4.2.** *Suppose that  $A$  is a continuous  $C_0(X)$ -algebra and  $\varphi$  is topologically free. A representation of the crossed product  $C^*(A, \alpha)$  is faithful if and only if it is faithful on  $A$ .*

**Corollary 4.3.** *If  $\varphi$  is free then every ideal in  $C^*(A, \alpha)$  is gauge-invariant.*

In order to use our pure infiniteness criterion - Proposition 2.46, we need to show that freeness of  $\varphi$  implies that  $A^+$  residually supports elements of  $C^*(A, \alpha)$ . To this end we use a technical device introduced in the following lemma, which will allow us to adapt the arguments of [14] to our setting. Recall that any  $C^*$ -algebra  $B$  is a  $M(B)$ -bimodule where  $(m \cdot b) := mb$  and  $(b \cdot m) := (m^*b^*)^*$ , for  $m \in M(B)$ ,  $b \in B$ .

**Lemma 4.4.** *The action of  $h \in C_0(X)$  on  $A$  as a multiplier of  $A$  extends to the action on  $C^*(A, \alpha)$  as a multiplier of  $C^*(A, \alpha)$  which is uniquely determined by the formulas*

$$(36) \quad h \cdot (au^n) := (h \cdot a)u^n, \quad (\alpha\alpha^n(b)u^n) \cdot h := au^n(b \cdot h) = \alpha\alpha^n(b \cdot h)u^n,$$

*where  $a, b \in A, n \in \mathbb{N}$ .*

*Proof.* Recall that  $A$  is a non-degenerate subalgebra of  $C^*(A, \alpha)$ . In other words, multiplication from the left defines a non-degenerate homomorphism from  $A$  into  $M(C^*(A, \alpha))$ . This homomorphism extends uniquely to the homomorphism from  $M(A)$  into  $M(C^*(A, \alpha))$ . Composing the latter with  $\mu_A : C_0(X) \rightarrow Z(M(A))$  we get a multiplier action of  $C_0(X)$  on  $C^*(A, \alpha)$  that clearly satisfies (36). In view of Proposition 2.14 formulas (36) determine this action uniquely.  $\square$

In the following statements we use the  $C_0(X)$ -bimodule structure on  $C^*(A, \alpha)$  described in the previous lemma (to increase readability we will suppress the symbol ‘ $\cdot$ ’).

**Lemma 4.5** (cf. Lemma 2.3 in [14]). *Let  $k > 0$  and  $a \in A\alpha^k(A)$ . Suppose that  $x_0 \in X$  is not fixed by  $\varphi^k$ . For every  $\varepsilon > 0$  there is  $h \in C_0(X)$  such that  $0 \leq h \leq 1$ ,  $h(x_0) = 1$  and  $\|h(au^k)h\| \leq \varepsilon$ .*

*Proof.* The proof of [14, Lemma 2.3] is readily adapted to our case; it suffices to replace the partial crossed product convolution formula with (36).  $\square$

**Proposition 4.6** (cf. Proposition 2.4 in [14]). *Suppose that either  $\varphi$  is topologically free and  $A$  is a continuous  $C_0(X)$ -algebra, or that  $\varphi$  is free. Then for every  $a \in C^*(A, \alpha)$  and every  $\varepsilon > 0$  there is  $h \in C_0(X)$  such that*

- (i)  $\|h\mathcal{E}(a)h\| \geq \|\mathcal{E}(a)\| - \varepsilon$ ,
- (ii)  $\|h\mathcal{E}(a)h - hah\| \leq \varepsilon$ ,
- (iii)  $h \geq 0$  and  $\|h\| = 1$ ,

where  $\mathcal{E}$  is the conditional expectation (10).

*Proof.* We adapt the proof of [14, Proposition 2.4]. A simple approximation argument implies that we may assume that  $a$  is of the form (11). Then  $\mathcal{E}(a) = a_0$ . Let us consider the non-empty set  $V = \{x \in X : \|a_0(x)\| > \|a_0\| - \varepsilon\}$  and notice that there exists  $x_0 \in V$  such that  $x_0$  is not a fixed point for  $\varphi^k$  for all  $k = 1, \dots, n$ . Indeed, if  $\varphi$  is free, existence of such  $x_0$  is obvious. If  $A$  is a continuous  $C_0(X)$ -algebra then  $V$  is open and the existence of  $x_0$  is guaranteed by topological freeness of  $\varphi$ . Applying Lemma 4.5 we see that for each  $k = \pm 1, \dots, \pm n$  there exists  $h_k \in C_0(X)$  such that

$$h_k(x_0) = 1, \quad \|h_k(a_k u^{|k|})h_k\| \leq \frac{\varepsilon}{2n}, \quad \text{and } 0 \leq h_k \leq 1.$$

Let  $h := \prod_{k=\pm 1, \dots, \pm n} h_k$ . Then (iii) is immediate, and (i) holds because  $\|ha_0h\| \geq \|a_0(x_0)\| > \|a_0\| - \varepsilon$ . For (ii), we have

$$\|ha_0h - hah\| \leq \sum_{k=\pm 1, \dots, \pm n} \|h(a_k u^{|k|})h\| \leq \sum_{k=\pm 1, \dots, \pm n} \|h_k(a_k u^{|k|})h_k\| < \varepsilon.$$

$\square$

**Proposition 4.7.** *If  $\varphi$  is free then  $A^+$  residually supports elements of  $C^*(A, \alpha)^+$ .*

*Proof.* Let  $\mathcal{I}$  be an ideal in  $C^*(A, \alpha)$ . By Corollaries 4.3 and 2.21, we have the isomorphism  $C^*(A, \alpha)/\mathcal{I} \cong C^*(A/I, \alpha_I)$  where  $I := A \cap \mathcal{I}$  is an invariant ideal in  $(A, \alpha)$ . The system  $(A/I, \alpha_I)$  is reversible by Lemma 2.16 ii). By Proposition 3.10,  $(A/I, \alpha_I)$  is induced by the morphism  $(\varphi, \{\alpha_{I,x}\}_{x \in \Delta})$ . Fix a positive element  $b$  in  $C^*(A, \alpha)/\mathcal{I}$ . Without loss of generality we may assume that  $\|b\| = 1$ . Applying Proposition 4.6 to  $(A/I, \alpha_I)$ , we may find a positive contraction  $h \in M(A/I)$  such that (21) holds. Now the last part of the proof of Proposition 2.42 shows that  $a := (h\mathcal{E}(b)h - 1/2)_+ \in A/I$  is non-zero element such that  $a \lesssim b$  relative to  $C^*(A, \alpha)/\mathcal{I} \cong C^*(A/I, \alpha_I)$ .  $\square$

**Corollary 4.8.** *Suppose that  $\varphi$  is free.*

- (i) *If  $A$  has the ideal property and is purely infinite, then the same holds  $C^*(A, \alpha)$ .*

- (ii) *If there are finitely many invariant ideals in  $(A, \alpha)$  and  $A$  is purely infinite, then  $C^*(A, \alpha)$  is purely infinite.*

*Proof.* The assertion follows from Proposition 4.7 and Proposition 2.46 modulo Remark 2.48.  $\square$

**4.2. Reversible extension of a  $C^*$ -dynamical system induced by a morphism.** Let us now get back to the case of a not necessarily reversible  $C_0(X)$ -dynamical system  $(A, \alpha)$ . Let  $(B, \beta)$  denote the reversible  $J$ -extension of  $(A, \alpha)$ , cf. Definition 2.30. We put

$$Y = \overline{\sigma_A(\text{Prim}(A/J))}.$$

In view of our standing assumption, equality (3) and the fact that  $J$  is contained in the set (30) we see that  $Y$  contains  $X \setminus \varphi(\Delta)$ . We denote by  $(\tilde{X}, \tilde{\varphi})$  the reversible  $Y$ -extension of the partial dynamical system  $(X, \varphi)$ , see Definition 2.32. Our aim is to use Proposition 3.15 to describe  $B$  as a  $C_0(\tilde{X})$ -algebra and  $\beta$  as an endomorphism induced by a morphism  $(\tilde{\varphi}, \{\beta_{\tilde{x}}\}_{\tilde{x} \in \Delta})$  for a certain field of homomorphisms  $\beta_{\tilde{x}}, \tilde{x} \in \Delta$ .

We start by fixing indispensable notation. Let  $\Delta_n := \varphi^{-n}(\Delta)$  be the domain of  $\varphi^n$ ,  $n \in \mathbb{N}$ . For  $x \in \Delta_n$  we put

$$\alpha_{(x,n)} := \alpha_x \circ \alpha_{\varphi(x)} \circ \dots \circ \alpha_{\varphi^{n-1}(x)}, \quad n > 0,$$

and  $\alpha_{(x,0)} := id$ . To each  $n \in \mathbb{N}$  and  $x \in X$  belonging to the domain of  $\varphi^n$ , we associate the hereditary subalgebra in  $A(x)$  generated by the range of  $\alpha_{(x,n)}$ :

$$(37) \quad A_n(x) := \alpha_{(x,n)}(A(\varphi^n(x)))A(x)\alpha_{(x,n)}(A(\varphi^n(x))).$$

We construct fibre  $C^*$ -algebras  $B(\tilde{x})$  as follows. If  $\tilde{x} = (x_0, x_1, \dots) \in X_\infty$ , we let

$$B(\tilde{x}) := \varinjlim \{A_n(x_n), \alpha_{x_{n+1}}\}$$

to be the inductive limit of the sequence  $A_0(x_0) \xrightarrow{\alpha_{x_1}} A_1(x_1) \xrightarrow{\alpha_{x_2}} A_2(x_2) \xrightarrow{\alpha_{x_3}} \dots$ . If  $\tilde{x} = (x_0, x_1, \dots, x_N, 0, \dots) \in X_N$ , we simply put

$$B(\tilde{x}) = A_N(x_N)/J(x_N).$$

In other words,  $B(\tilde{x}) = q_{x_N}(A_N(x_N))$  where  $(id, \{q_x\}_{x \in Y})$  is the morphism that induces the quotient map  $q : A \rightarrow A/J$ , see Example 3.6.

We will represent the dense  $*$ -subalgebra  $\bigcup_{n \in \mathbb{N}} B_n$  of  $B$  as an algebra of sections of  $\mathcal{B} = \bigsqcup_{\tilde{x} \in \tilde{X}} B(\tilde{x})$ . For every  $a = (a_0 + J) \oplus \dots \oplus (a_{n-1} + J) \oplus a_n \in B_n$  we define the section  $\pi(\phi_n(a))$  of  $\mathcal{B}$  by the formula

$$\pi(\phi_n(a))(\tilde{x}) = \begin{cases} a_N(x_N) + J(x_N), & \tilde{x} \in X_N, N \leq n, \\ \alpha_{(x_N, N-n)}(a_n(x_n)) + J(x_N), & \tilde{x} \in X_N, N > n, \\ [\underbrace{0, \dots, 0}_n, a_n(x_n), \alpha_{x_{n+1}}(a_n(x_n)), \alpha_{(x_{n+2}, 2)}(a(x_n)), \dots], & \tilde{x} \in X_\infty. \end{cases}$$

We define endomorphisms  $\beta_{\tilde{x}} : B(\tilde{\varphi}(\tilde{x})) \rightarrow B(\tilde{x})$ ,  $\tilde{x} \in \tilde{\Delta}$ , as follows. We put

$$\beta_{\tilde{x}}[a_0, a_1, a_2, \dots] := [a_1, a_2, \dots], \quad \text{if } \tilde{x} \in X_\infty \cap \tilde{\Delta},$$

and if  $\tilde{x} \in X_N \cap \tilde{\Delta}$  we let  $\beta_{\tilde{x}}$  be the inclusion map corresponding to the inclusion  $B(\tilde{\varphi}(\tilde{x})) = q_{x_N}(A_{N+1}(x_N)) \subseteq B(\tilde{x}) = q_{x_N}(A_N(x_N))$ .

If the system  $(A, \alpha)$  happens to be extendible, then by Lemma 3.12 the homomorphisms  $\alpha_x$ , and hence also  $\alpha_{(x,n)}$ , are extendible. In this case we put

$$p_{(x,0)} := 1_x, \quad x \in X, \quad \text{and} \quad p_{(x,n)} := \overline{\alpha}_{(x,n)}(1_{\varphi^n(x)}) \quad \text{for } n > 0, \quad x \in \varphi^{-n}(\Delta).$$

With this notation the algebras  $A_n(x)$  are corners  $p_{(x,n)}A(x)p_{(x,n)}$ . We define positive linear maps  $\beta_{*,\tilde{\varphi}(\tilde{x})} : B(\tilde{x}) \rightarrow B(\tilde{\varphi}(\tilde{x}))$ ,  $\tilde{x} \in \tilde{\Delta}$ , as follows. For  $X_\infty \cap \tilde{\Delta}$  we set

$$\beta_{*,\tilde{\varphi}(\tilde{x})}[a_0, a_1, a_2, \dots] := [0, p_{(x_0,1)} a_0 p_{(x_0,1)}, p_{(x_1,2)} a_1 p_{(x_1,2)}, \dots].$$

If  $\tilde{x} \in X_N \cap \tilde{\Delta}$ , we put  $\beta_{*,\tilde{\varphi}(\tilde{x})}(q_{x_N}(a)) := q_{x_N}(p_{(x_N,N+1)} a p_{(x_N,N+1)})$ .

**Theorem 4.9.** *Retain the above notation. There is a unique topology on  $\mathcal{B} = \bigsqcup_{\tilde{x} \in \tilde{X}} B(\tilde{x})$  making it into an upper semicontinuous  $C^*$ -bundle over  $\tilde{X}$  such that  $\pi$  establishes the isomorphism  $B \cong \Gamma_0(\mathcal{B})$ . Identifying  $B$  with the algebra of continuous sections of  $\mathcal{B}$  we have*

$$\beta(a)(\tilde{x}) = \begin{cases} \beta_{\tilde{x}}(a(\tilde{\varphi}(\tilde{x}))), & \tilde{x} \in \tilde{\Delta}, \\ 0, & \tilde{x} \notin \tilde{\Delta}, \end{cases}$$

and if  $(A, \alpha)$  is extendible, then  $(B, \beta)$  is extendible and

$$\beta_*(a)(\tilde{x}) = \begin{cases} \beta_{*,\tilde{x}}(a(\tilde{\varphi}^{-1}(\tilde{x}))), & \tilde{x} \in \tilde{\varphi}(\tilde{\Delta}), \\ 0, & \tilde{x} \notin \tilde{\varphi}(\tilde{\Delta}), \end{cases}$$

where  $\beta_*$  is the unique regular transfer operator for  $(B, \beta)$ . Moreover, if  $A$  is a continuous  $C_0(X)$ -algebra and either

- (i)  $J$  is a complemented ideal, every ideal  $J(x)$  is trivial (i.e. either  $\{0\}$  or  $A(x)$ ), and every homomorphism  $\alpha_x$  is injective, or
- (ii)  $\sigma_A$  is injective, i.e.  $X \cong \text{Prim}(A)$ , cf. Example 3.8,

then  $\mathcal{B} = \bigsqcup_{\tilde{x} \in \tilde{X}} B(\tilde{x})$  is a continuous  $C^*$ -bundle.

*Proof.* Notice that  $B_n$ ,  $n \in \mathbb{N}$ , is naturally a  $C_0(Y_n)$ -algebra with

$$Y_n := Y \sqcup Y \cap \Delta_1 \sqcup \dots \sqcup Y \cap \Delta_{n-1} \sqcup \Delta_n$$

where  $\sqcup$  denotes the disjoint sum of topological spaces. Moreover, the bonding homomorphism  $\alpha_n : B_n \rightarrow B_{n+1}$  is induced by a morphism. Indeed, consider the map  $\varphi_n : Y_{n+1} \rightarrow Y_n$  given by the diagram

$$\begin{array}{ccccccc} Y_n & = & Y & \sqcup & \dots & \sqcup & Y \cap \Delta_{n-1} & \sqcup & \Delta_n \\ \varphi_n \uparrow & & id \uparrow & & & & id \uparrow & & id \uparrow & \swarrow \varphi \\ Y_{n+1} & = & Y & \sqcup & \dots & \sqcup & Y \cap \Delta_{n-1} & \sqcup & Y \cap \Delta_n & \sqcup & \Delta_{n+1} \end{array}$$

Let  $y \in Y_{n+1}$ . Define  $\alpha_{x,n} : B_n(\varphi_n(y)) \rightarrow B_{n+1}(y)$  to be identity if  $y \in Y \cap \Delta_k$  belongs to the  $k$ -th summand of  $Y_{n+1}$ ,  $k \leq n-1$ , to be  $q_y$  if  $y \in Y \cap \Delta_n$  belongs to the  $n$ -th summand of  $Y_{n+1}$ , and to be  $\alpha_y$  if  $y \in \Delta_{n+1}$  belongs to the last summand of  $Y_{n+1}$ . Then  $\alpha_n : B_n \rightarrow B_{n+1}$  is induced by  $(\varphi_n, \{\alpha_{x,n}\}_{x \in Y_{n+1}})$ .

Since  $\varphi(\Delta) \cup Y = X$ , the mappings  $\varphi_n : Y_{n+1} \rightarrow Y_n$  are surjective and we may apply Proposition 3.15 to the inductive system  $\{B_n, \alpha_n\}_{n \in \mathbb{N}}$ . The arising direct limit  $C^*$ -bundle can be identified with the one described above. Indeed, we have a natural homeomorphism  $\Phi : \tilde{X} \rightarrow \varprojlim \{Y_{n+1}, \varphi_n\}$ . Namely, for  $\tilde{x} = (x_0, x_1, x_2, \dots) \in X_\infty$  we define  $\Phi(\tilde{x}) = (x_0, x_1, x_2, \dots) \in \varprojlim \{Y_{n+1}, \varphi_n\}$  where in the latter we treat  $x_n \in \Delta_n$  as the point in last direct summand of  $Y_n$  for all  $n \in \mathbb{N}$ . For  $\tilde{x} = (x_0, x_1, \dots, x_N, 0, 0, \dots) \in X_N$  we define  $\Phi(\tilde{x}) = (x_0, x_1, \dots, x_N, x_N, x_N, \dots) \in \varprojlim \{Y_{n+1}, \varphi_n\}$  where in the latter we treat  $x_n \in \Delta_n$  as the point in the last direct summand of  $Y_n$ , for  $n \leq N$ , and  $x_N \in Y \cap \Delta_N$  as the point in  $N$ -th direct summand in  $Y_n$  for  $n \geq N$ . For  $\tilde{x} \in X_\infty$ , the algebras  $B(\tilde{x})$  and  $B(\Phi(\tilde{x}))$  are naturally isomorphic because they arise as direct limits of direct sequences that can be naturally identified. For  $\tilde{x} = (x_0, x_1, \dots, x_N, 0, 0, \dots) \in X_N$ ,  $B(\Phi(\tilde{x}))$  is naturally isomorphic to  $B_N(x_N)$  where  $x_N \in Y_N$  lies in the last direct summand of  $Y$ . Thus  $B(\Phi(\tilde{x}))$  is naturally isomorphic to  $A_N(x_N)/J(x_N)$ . Hence we may identify the corresponding fibers of bundles. Then we get  $\pi(\phi_n(a))(\tilde{x}) = \phi_n(a)(\Phi(\tilde{x}))$  for any  $a \in B_n$  and  $\tilde{x} \in \tilde{X}$ . This proves the first part of the assertion.

Let  $a = a_0 + J \oplus \dots \oplus a_{n-1} + J \oplus a_n \in B_n$  and  $\tilde{x} \in \tilde{X}$ . Note that  $\pi(\phi_n(\beta_n(a))) (\tilde{x})$  is equal to

$$\begin{cases} a_{N+1}(x_N) + J(x_N), & \tilde{x} \in X_N, \ N+1 \leq n, \\ \alpha_{(x_N, N+1-n)}(a_n(x_{n-1})) + J(x_N), & \tilde{x} \in X_N, \ N+1 > n, \\ \underbrace{[0, \dots, 0]}_n, \alpha_{x_n}(a_n(x_{n-1})), \alpha_{(x_{n+1}, 1)}(a(x_{n-1})), \dots, & \tilde{x} \in X_\infty. \end{cases}$$

Suppose first that  $\tilde{x} = (x_0, \dots, x_n, \dots) \notin \tilde{\Delta}$ . Since  $x_0 \in X \setminus \Delta$  we either have  $x_N \in \Delta_N \setminus \Delta_{N+1}$ , when  $\tilde{x} \in X_N$  for  $N < n$ , or  $x_n \in \Delta_n \setminus \Delta_{n+1}$ , otherwise. Thus  $\pi(\beta_n(a))(\tilde{x}) = 0$  because for any  $x \notin \Delta_k$ , using that  $a_k \in \alpha^k(A)A\alpha^k(A)$ , we get  $a_k(x) = 0$ . On the other hand, for  $\tilde{x} \in \tilde{\Delta}$  one sees that

$$\pi(\phi_n(\beta_n(a))) (\tilde{x}) = \beta_{\tilde{x}}(\pi(\phi_n(a))(\tilde{\varphi}(\tilde{x}))).$$

Hence, in view of Proposition 3.2,  $\beta : B \rightarrow B$  is induced by the morphism  $(\tilde{\varphi}, \{\beta_{\tilde{x}}\}_{\tilde{x} \in \tilde{\Delta}})$ .

If  $(A, \alpha)$  is extendible then  $(B, \beta)$  is extendible by [33, Proposition 2.4]. Moreover, invoking Proposition 3.13 and formula (32) one concludes that the corresponding transfer operator  $\beta_*$  satisfies the formula described in the assertion with the mappings

$$\beta_{*, \tilde{\varphi}(\tilde{x})}(b) := \beta_{\tilde{x}}^{-1} \left( \overline{\beta_{\tilde{x}}}(1_{\tilde{\varphi}(\tilde{x})}) a \overline{\beta_{\tilde{x}}}(1_{\tilde{\varphi}(\tilde{x})}) \right), \quad b \in B(\tilde{x}), \ \tilde{x} \in \tilde{\Delta},$$

where  $\beta_{\tilde{x}}^{-1}$  is the inverse to the isomorphism  $\beta_{\tilde{x}} : (\ker \beta_{\tilde{x}})^\perp \rightarrow \overline{\beta_{\tilde{x}}}(1_{\tilde{\varphi}(\tilde{x})})B(\tilde{x})\overline{\beta_{\tilde{x}}}(1_{\tilde{\varphi}(\tilde{x})})$ . We leave it to the reader to check that these maps coincide with the maps we have previously described. This proves the second part of the assertion.

For the last part of the assertion it suffices to apply the second part of Proposition 3.15. Indeed, if all ideals  $J(x)$  are trivial and all homomorphisms  $\alpha_x$  are injective, then all homomorphisms  $\alpha_{x,n}$  are injective. Moreover, if  $J$  is complemented or  $\sigma_A$  is injective, then  $B_n$ ,  $n \in \mathbb{N}$ , is a continuous  $C_0(Y_n)$ -algebra by the last part of Lemma 1.7.  $\square$

**Remark 4.10.** The morphism  $(\tilde{\varphi}, \{\beta_{\tilde{x}}\}_{\tilde{x} \in \tilde{\Delta}})$  constructed above can be considered canonical. In particular,  $\tilde{\varphi}$  is always a partial homeomorphism and all the homomorphisms  $\beta_{\tilde{x}}$  are injective. Thus even when the initial system  $(A, \alpha)$  is already reversible and  $J = (\ker \alpha)^\perp$ ,



so that we have  $(A, \alpha) = (B, \beta)$ , the morphism  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  may differ from  $(\tilde{\varphi}, \{\beta_{\tilde{x}}\}_{\tilde{x} \in \tilde{\Delta}})$ . For instance, for the reversible dynamical system  $(A, \alpha)$  described in Example 3.14 we obtain (omitting zero fibers) that  $B = A = \mathbb{C}^3$  is naturally a  $C_0(\{1, 2, 3\})$ -algebra and  $\beta = \alpha$  is induced by the morphism  $(\varphi, \{\alpha_1, \alpha_3\})$  where  $\varphi(1) = 2$ ,  $\varphi(3) = 3$ ,  $\alpha_1 = \alpha_3 = id$ .

**4.3. General results.** Now, we put together the results of the previous subsections. For the first statement recall Definition 2.37.

**Theorem 4.11.** *Let  $(A, \alpha)$  be a continuous  $C_0(X)$ -dynamical system. Suppose that  $\varphi$  is topologically free outside the set  $Y = \overline{\sigma_A(\text{Prim}(A/J))}$  and either (i) or (ii) in Theorem 4.9 holds. Every injective representation  $(\pi, U)$  of  $(A, \alpha)$  such that  $J = \{a \in A : U^*U\pi(a) = \pi(a)\}$  give rise to a faithful representation  $\pi \rtimes U$  of  $C^*(A, \alpha; J)$ .*

*Proof.* By Theorem 4.9 the reversible  $J$ -extension  $(B, \beta)$  of  $(A, \alpha)$  is induced by a morphism based on the reversible  $Y$ -extension  $(\tilde{X}, \tilde{\varphi})$  of  $(X, \varphi)$ . Moreover,  $B$  is a continuous  $C_0(\tilde{X})$ -algebra and  $\tilde{\varphi}$  is topologically free by Lemma 2.38. Hence the assertion follows from Corollary 4.2 applied to  $(B, \beta)$ .  $\square$

The first part of the following result also implies the uniqueness property described in Theorem 4.11. It does not require continuity of the  $C_0(X)$ -algebra  $A$ , still it requires freeness of  $\varphi$  which is a much stronger condition than topological freeness.

**Theorem 4.12.** *Suppose that  $\varphi$  is free. Then all ideals in  $C^*(A, \alpha; J)$  are gauge-invariant and, in particular, they are in one-to-one correspondence with  $J$ -pairs for  $(A, \alpha)$ . Moreover,*

- (i) *If  $A$  has the ideal property and is purely infinite, then the same holds  $C^*(A, \alpha; J)$ .*
- (ii) *If there are finitely many  $J$ -pairs for  $(A, \alpha)$  and  $A$  is purely infinite, then  $C^*(A, \alpha; J)$  has finitely many ideals and is purely infinite.*

*Proof.* Let  $(B, \beta)$  be the natural  $J$ -extension of  $(A, \alpha)$ . We show that  $(B, \beta)$  satisfies the assumptions of Corollary 4.8, when treated as induced by the morphism  $(\tilde{\varphi}, \{\beta_{\tilde{x}}\}_{\tilde{x} \in \tilde{\Delta}})$  described in Theorem 4.9. Plainly,  $\tilde{\varphi}$  is free, cf. Lemma 2.38. Hence the first part of the assertion follows by Corollary 4.3, applied to  $(B, \beta)$ .

Suppose that  $A$  is purely infinite. Since pure infiniteness is preserved under taking direct sums, quotients, hereditary subalgebras and direct limits, see [27, Propositions 4.3, 4.17 and 4.18], we conclude that  $B$  is purely infinite.

(i). It is easy to see that the ideal property is preserved under taking direct sums and quotients. It is also preserved when passing to direct limits [45, Proposition 2.2] and in the presence of pure infiniteness it also passes to hereditary subalgebras, see [47, Proposition 2.10]. Thus  $B$  is purely infinite and has the ideal property. Accordingly, Corollary 4.8 (i) applies.

(ii). By Lemma 2.36 (ii) there are finitely many invariant ideals in  $(B, \beta)$ . Hence Corollary 4.8 (ii) applies to  $(B, \beta)$ .  $\square$

## 5. CROSSED PRODUCTS OF $C^*$ -ALGEBRAS WITH HAUSDORFF PRIMITIVE IDEAL SPACE

In this section, we fix a  $C^*$ -algebra with a Hausdorff primitive ideal space  $X = \text{Prim}(A)$  and consider a  $C_0(X)$ -dynamical system  $(A, \alpha)$  described in Example 3.8. Let  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  be the morphism determining  $\alpha$ . By Lemma 3.4, without loss of generality we may assume

that every  $\alpha_x$ ,  $x \in \Delta$ , is nonzero (and thus injective), so that  $(\varphi, \{\alpha_x\}_{x \in \Delta})$  is uniquely determined by  $\alpha$ . Thus, we make the following standing assumptions:

- $X = \text{Prim}(A)$  is a Hausdorff space,  $\sigma_A = \text{id}$ , and every  $\alpha_x$ ,  $x \in \Delta$ , is non-zero.

In particular, we have a bijective correspondence

$$(38) \quad X \supseteq V \longmapsto I_V := \{a \in A : a(x) = 0 \text{ for all } x \in V\} \triangleleft A$$

between closed subsets of  $X$  and ideals in  $A$ . We use it to describe ideal structure of the crossed product  $C^*(A, \alpha; J)$  in terms of the dynamical system  $(X, \varphi)$ . We also give some criteria for  $C^*(A, \alpha; J)$  to be purely infinite or a Kirchberg algebra. We finish this section by describing the  $K$ -theory of all ideals and quotients in  $C^*(A, \alpha; J)$  when  $A = C_0(X, D)$  with  $D$  a simple  $C^*$ -algebra.

**5.1. Ideal structure of  $C^*(A, \alpha; J)$ .** In this subsection, we generalize results proved in the commutative case (i.e., when  $D = \mathbb{C}$ ) in [33, Subsection 4.6]. Let us fix an ideal  $J$  in  $(\ker \alpha)^\perp$ . Since  $(\ker \alpha)^\perp = \overline{I_{X \setminus \varphi(\Delta)}}$  we have

$$J = I_Y \text{ where } Y \text{ is a closed subset of } X \text{ such that } Y \cup \varphi(\Delta) = X.$$

We have the following dual version of Definitions 2.15 and 2.17.

**Definition 5.1** (Definition 4.9 in [33]). A closed set  $V \subseteq X$  is *positively invariant* under  $\varphi$  if  $\varphi(V \cap \Delta) \subseteq V$ , and  $V$  is  *$Y$ -negatively invariant* if  $V \subseteq Y \cup \varphi(V \cap \Delta)$ . If  $V$  is both *positively* and  *$Y$ -negatively invariant*, we call it  *$Y$ -invariant*. We say that  $V$  is *invariant* if it is  $\overline{X \setminus \varphi(\Delta)}$ -invariant. A pair  $(V, V')$  of closed subsets of  $X$  satisfying

$$V \text{ is positively } \varphi\text{-invariant, } V' \subseteq Y \text{ and } V' \cup \varphi(V \cap \Delta) = V$$

is called a  $Y$ -pair for  $(X, \varphi)$ .

**Lemma 5.2.** *An ideal  $I_V$  in  $A$  is positively (resp.  $J$ -)invariant if and only if  $V$  is positively (resp.  $Y$ -)invariant. A pair  $(I_V, I_{V'})$  of ideals in  $A$  is a  $J$ -pair for  $(A, \alpha)$  if and only if  $(V, V')$  is a  $Y$ -pair for  $(X, \varphi)$ .*

*Proof.* We recall that  $\varphi : \Delta \rightarrow X$  is necessarily a closed map. In particular, if  $V$  is closed then  $\varphi(V \cap \Delta)$  is also closed. Since the endomorphisms  $\alpha_x$ ,  $x \in \Delta$ , are injective, one readily sees that  $\alpha^{-1}(I_V) = I_{\varphi(V \cap \Delta)}$ . Using this observation we get the following equivalences

$$\alpha(I_V) \subseteq I_V \iff I_V \subseteq \alpha^{-1}(I_V) \iff \varphi(V \cap \Delta) \subseteq V,$$

$$J \cap \alpha^{-1}(I_V) \subseteq I_V \iff I_{Y \cup \varphi(V \cap \Delta)} \subseteq I_V \iff V \subseteq Y \cup \varphi(V \cap \Delta),$$

This proves the initial part of the assertion. Similarly as above, we get

$$I_{V'} \cap \alpha^{-1}(I_V) = I_V \iff V' \cup \varphi(V \cap \Delta) = V.$$

Since  $I_{V'} \subseteq J$  if and only if  $V' \subseteq Y$ , this completes the proof.  $\square$

We note that the class  $C^*$ -dynamical systems satisfying our standing assumptions is closed under taking quotients and restrictions.

**Lemma 5.3.** *If  $V$  is a positively invariant closed set, then the quotient endomorphism  $\alpha_{I_V}$  is induced by the morphism  $(\varphi|_{\Delta \cap V}, \{\alpha_x\}_{x \in \Delta \cap V})$  where we treat  $A/I_V$  as a  $C_0(V)$ -algebra, and the restricted endomorphism  $\alpha|_{I_V}$  is induced by the morphism  $(\varphi|_{\Delta \setminus \varphi^{-1}(V)}, \{\alpha_x\}_{x \in \Delta \setminus \varphi^{-1}(V)})$  where we treat  $I_V$  as a  $C_0(X \setminus V)$ -algebra.*

*Proof.* It readily follows from Proposition 3.11.  $\square$

Now we describe the dual topological version of the system introduced in Definition 2.23, it's action is schematically presented in [30, Figure 1].

**Definition 5.4.** We define a partial dynamical system  $(X^Y, \varphi^Y)$  by putting

$$X^Y := \varphi(\Delta) \sqcup Y, \quad \Delta^Y := (\varphi(\Delta) \cap \Delta) \sqcup (Y \cap \Delta) \subseteq X^Y$$

and letting  $\varphi^Y : \Delta^Y \rightarrow X^Y$  to map a point  $x$  from  $\Delta^Y$  to the point  $\varphi(x)$  lying in the first disjoint summand  $\varphi(\Delta)$  of  $X^Y$ .

**Lemma 5.5.** *Let  $(A^J, \alpha^J)$  be the  $C^*$ -dynamical system described in Definition 2.23. We may assume the identification  $\text{Prim}(A^J) = X^Y$ , and treating  $A^J$  as a  $C_0(X^Y)$ -algebra the endomorphism  $\alpha^J$  is induced by the morphism  $(\varphi^Y, \{\alpha_x\}_{x \in \Delta^Y})$ . Moreover, if  $(V, V')$  is  $Y$ -pair for  $(X, \varphi)$ , then*

$$(39) \quad (V, V')^Y := (\varphi(\Delta) \cap V) \sqcup V' \subseteq X^Y$$

*is a closed invariant set in  $(X^Y, \varphi^Y)$  that corresponds to a positively invariant ideal in  $(A^J, \alpha^J)$  given by*

$$(40) \quad I_{(V, V')^Y} = \{a \in A^J : a(x) = 0 \text{ for } x \in (V, V')^Y\}.$$

*Proof.* In particular, assuming the identification  $\text{Prim}(A^J) = X^Y$ , cf. Lemma 5.5, for the corresponding  $Y$ -pair  $(V, V')$  we have  $(I_V, I_{V'})^J = q_{\ker \alpha}(I_V) \oplus q_J(I_{V'})$ . Thus the assertion follows by Proposition 2.25.  $\square$

The following proposition generalizes [33, Proposition 4.9] (proved in the commutative case) and in addition it describes up to Morita-Rieffel equivalence all ideals in  $C^*(A, \alpha; J)$ .

**Proposition 5.6.** *If  $\varphi$  is free then all ideals in  $C^*(A, \alpha; J)$  are gauge-invariant. In general, we have a bijective correspondence between gauge-invariant ideals  $\mathcal{I}$  in  $C^*(A, \alpha; J)$  and  $Y$ -pairs  $(V, V')$  for  $(X, \varphi)$  established by relations*

$$I_V = A \cap \mathcal{I}, \quad I_{V'} = \{a \in A : (1 - u^*u)a \in \mathcal{I}\}.$$

*Moreover, for the corresponding objects we have an isomorphism*

$$(41) \quad C^*(A, \alpha; J)/\mathcal{I} \cong C^*(A/I_V, \alpha_{I_V}; q_{I_V'}(I_V')),$$

*and if  $V' = V \cap Y$  (equivalently if  $\mathcal{I}$  is generated by  $I_V$ ), then  $\mathcal{I}$  is Morita-Rieffel equivalent to  $C^*(I_V, \alpha|_{I_V}; I_{V \cup Y})$ . In general,  $\mathcal{I}$  is Morita-Rieffel equivalent to*

$$C^*(I_{(V, V')^Y}, \alpha^J|_{I_{(V, V')^Y}, D})$$

*where  $I_{(V, V')^Y}$  is given by (40) and  $\alpha^J : A^J \rightarrow A^J$  is induced by the morphism  $(\varphi^Y, \{\alpha_x\}_{x \in \Delta^Y})$ .*

*Proof.* By Theorem 4.12 every ideal in  $C^*(A, \alpha; J)$  is gauge-invariant. Hence the assertion follows from Theorem 2.19 and Lemmas 5.2 and 5.5.  $\square$

**Corollary 5.7.** *Suppose that  $\varphi$  is free and that  $\varphi(\Delta)$  is open in  $X$  (equivalently  $\ker \alpha$  is a complemented ideal in  $A$ ). We have a bijective correspondence between ideals  $\mathcal{I}$  in  $C^*(A, \alpha)$  and invariant sets  $V$  for  $(X, \varphi)$  established by the relation  $I_V = A \cap \mathcal{I}$ . Moreover, for every ideal  $\mathcal{I}$  and the corresponding invariant set  $V$  we have an isomorphism*

$$C^*(A, \alpha)/\mathcal{I} \cong C^*(A/I_V, \alpha_{I_V})$$

and  $\mathcal{I}$  is Morita-Rieffel equivalent to  $C^*(I_V, \alpha|_{I_V})$ .

*Proof.* In the proof of Proposition 5.6, instead of Theorem 2.19 we may apply Corollary 2.21.  $\square$

**5.2. Pure infiniteness and simplicity.** For separable  $C^*$ -algebras we have the following result concerning permanence of pure infiniteness and the ideal property.

**Proposition 5.8.** *Suppose that  $A$  is separable and purely infinite and assume that  $\varphi$  is free. If either  $X$  is totally disconnected or there are finitely many  $Y$ -pairs for  $(X, \varphi)$ , then  $C^*(A, \alpha; J)$  is purely infinite and has the ideal property.*

*Proof.* If  $X$  is totally disconnected, then  $A$  has the ideal property by [27, Proposition 2.11]. Thus  $C^*(A, \alpha; J)$  is purely infinite and has the ideal property by Theorem 4.12 (i). If there are finitely many  $Y$ -pairs for  $(X, \varphi)$ , then  $C^*(A, \alpha; J)$  is purely infinite and has finitely many ideals by Theorem 4.12 (ii). Hence [27, Proposition 2.11] implies that  $C^*(A, \alpha; J)$  has the ideal property.  $\square$

The following characterization of simplicity of  $C^*(A, \alpha)$ , cf. Remark 2.20, is a far reaching generalization of [33, Theorem 4.4], proved in the case  $A$  is commutative.

**Proposition 5.9.** *The crossed product  $C^*(A, \alpha)$  is simple if and only one of the two possible cases hold:*

- (i)  $X$  is discrete and  $(X, \varphi)$  is (up to conjugacy) either a truncated shift on  $\{1, \dots, n\}$ , one sided shift on  $\mathbb{N}$ , or a two-sided shift on  $\mathbb{Z}$ ,
- (ii)  $X$  is not discrete and  $\varphi : X \rightarrow X$  is a surjection such that there are no non-trivial closed subsets  $V$  of  $X$  satisfying  $\varphi(V) = V$ .

*Proof.* If (i) or (ii) holds then there are no non-trivial closed invariant set  $V$  in  $(X, \varphi)$  and  $\varphi$  is free. Hence  $C^*(A, \alpha)$  is simple by Corollary 5.7.

Conversely, suppose that  $C^*(A, \alpha)$  is simple. By Proposition 5.6 there are no non-trivial closed invariant sets in  $(X, \varphi)$ . The argument in the proof of [33, Theorem 4.4] shows that either (i) or (ii) holds. In particular, if  $\varphi : X \rightarrow X$  is a surjection then a closed set  $V$  is invariant in  $(X, \varphi)$  if and only if  $\varphi(V) = V$ .  $\square$

**Corollary 5.10.** *Suppose that  $A$  is purely infinite, nuclear and separable. The crossed product  $C^*(A, \alpha)$  is a Kirchberg  $C^*$ -algebra if and only if one of the conditions (i) or (ii) in Proposition 5.9 is satisfied.*

*If  $C^*(A, \alpha)$  is a Kirchberg  $C^*$ -algebra, and  $A$  satisfies the UCT then  $C^*(A, \alpha)$  satisfies the UCT.*

*Proof.* Plainly,  $C^*(A, \alpha; J)$  is separable, and it is nuclear by Proposition 2.10 (ii). Proposition 5.8 implies that  $C^*(A, \alpha; J)$  is purely infinite if one of the conditions in Proposition 5.9 holds. Hence the assertion follows from Proposition 5.9.

Now suppose that  $C^*(A, \alpha)$  is a Kirchberg  $C^*$ -algebra and  $A$  satisfies the UCT. If  $\varphi$  is surjective then  $(\ker \alpha)^\perp = A$  and if  $\varphi$  is not surjective then  $X \setminus \{\varphi(\Delta)\} = \{x_0\}$ , cf. Proposition 5.9 (i), and  $A = (\ker \alpha)^\perp \oplus A(x_0)$ . In both cases  $(\ker \alpha)^\perp$  satisfies the UCT and thus  $C^*(A, \alpha)$  satisfies the UCT by Proposition 2.10 (iii).  $\square$

**5.3.  $K$ -theory in the case of a trivial bundle.** In this subsection, we assume that the associated  $C^*$ -bundle  $\mathcal{A}$  is trivial. In other words, we assume that  $A = C_0(X, D)$  where  $D$  is a simple  $C^*$ -algebra. By Proposition 3.9,  $\alpha$  is given by the formula (27) where  $\varphi : \Delta \rightarrow X$  is proper continuous map defined on an open subset  $\Delta \subseteq X$ , and  $\Delta \ni x \mapsto \alpha_x \in \text{End}(D) \setminus \{0\}$  is a continuous map. Actually, we make the following standing assumptions:

- $A = C_0(X, D)$  where  $D$  is a simple  $C^*$ -algebra,
- $X$  is totally disconnected,  $G := K_0(D)$  is torsion free and  $K_1(D) = \{0\}$ .

We treat  $G$  as a discrete group and denote by  $C_0(X, G)$  the set of continuous functions  $f : X \rightarrow G$  such that  $f^{-1}(G \setminus \{0\})$  is compact. In other words, any  $f \in C_0(X, G)$  is of the form  $f = \sum_{i=1}^n \chi_{X_i} \tau_i$  where  $X_i$ 's are compact and open subsets of  $X$  and  $\tau_i \in G$ . We consider  $C_0(X, G)$  an abelian group with the group operation defined pointwise. We also put  $C_0(\emptyset, G) := \{0\}$ .

**Lemma 5.11.** *For each  $\tau \in G$ , the function  $\Delta \ni x \mapsto K_0(\alpha_x)(\tau) \in G$  is continuous.*

*Proof.* For any projection  $p$  in  $M_n(D)$ , the function  $x \mapsto \alpha_x(p) \in M_n(D)$  is continuous. Hence the function  $x \mapsto [\alpha_x(p)]_0 \in G$  is locally constant. This implies the assertion.  $\square$

**Definition 5.12.** Let  $\delta_\alpha$  be a group homomorphism  $\delta_\alpha : C_0(X, G) \rightarrow C_0(X, G)$  given by

$$\delta_\alpha(f)(x) = \begin{cases} f(x) - K_0(\alpha_x)(f(\varphi(x))), & x \in \Delta \\ 0 & x \notin \Delta. \end{cases}$$

Note that  $\delta_\alpha$  is well defined by Lemma 5.11. We define  $\delta_\alpha^Y : C_0(X \setminus Y, G) \rightarrow C_0(X, G)$  to be the restriction of  $\delta_\alpha$ . We put

$$K_0(X, \varphi, \{\alpha_x\}_{x \in \Delta}; Y) := \text{coker}(\delta_\alpha^Y), \quad K_1(X, \varphi, \{\alpha_x\}_{x \in \Delta}; Y) := \ker(\delta_\alpha^Y).$$

If  $Y = X \setminus \varphi(\Delta)$  then we write  $K_i(X, \varphi, \{\alpha_x\}_{x \in \Delta}) := K_i(X, \varphi, \{\alpha_x\}_{x \in \Delta}; Y)$ ,  $i = 0, 1$ .

**Proposition 5.13.** *We have the following isomorphism*

$$(42) \quad K_*(C^*(C_0(X, D), \alpha; J)) \cong K_*(X, \varphi, \{\alpha_x\}_{x \in \Delta}; Y).$$

*Proof.* Since  $G = K_0(D)$  is torsion free,  $K_1(C_0(X)) = 0$  and  $K_1(D) = 0$ , by Künneth formulas, see for instance [53, Proposition 2.11], we get

$$K_0(C_0(X, D)) \cong K_0(C_0(X)) \otimes G, \quad K_0(C_0(X, D)) = \{0\},$$

where the isomorphism  $\Psi : K_0(C_0(X, D)) \rightarrow K_0(C_0(X)) \otimes K_0(D)$  is determined by the natural identifications

$$M_r(C_0(X) \otimes D) = C_0(X) \otimes M_r(D), \quad r \in \mathbb{N}.$$

It is well known, that the maps  $\text{Proj}(M_r(C_0(X))) \ni p \mapsto \text{Tr} \circ p \in C_0(X, \mathbb{Z})$  determine the isomorphism  $K_0(C_0(X)) \cong C_0(X, \mathbb{Z})$ , cf. [51, Exercise 3.4]. The formula

$$C_0(X, \mathbb{Z}) \otimes G \ni f \otimes \tau \mapsto f_\tau \in C_0(X, G), \text{ where } f_\tau(x) := f(x)\tau, x \in X,$$

determines an isomorphism  $\Phi : C_0(X, \mathbb{Z}) \otimes G \rightarrow C_0(X, G)$ , and to see it is enough to note that any element in  $C_0(X, \mathbb{Z}) \otimes G$  can be presented as a sum of the form  $\sum_{i=1}^n \chi_{X_i} \otimes \tau_i$  where  $X_i$ 's are compact-open and pairwise disjoint subsets of  $X$  and  $\tau_i \in G$ .

Composing the aforementioned isomorphisms we conclude that we have the isomorphism

$$K_0(C_0(X, D)) \cong C_0(X, G)$$

whose inverse sends a function  $f = \sum_{i=1}^n \chi_{X_i}[p_i]_0 \in C_0(X, G)$ , with  $X_i$  compact-open and disjoint, and  $p_i \in \text{Proj}(\otimes M_r(D))$ , to the element  $[\sum_{i=1}^n \chi_{X_i} p_i]_0 \in K_0(C_0(X, D))$ . We recall that  $A = C_0(X) \otimes D$  and  $J = C_0(X \setminus Y) \otimes D$ . The above analysis shows that the following diagram

$$\begin{array}{ccc} K_0(J) & \xrightarrow{K_0(\iota) - K_0(\alpha|_J)} & K_0(A) \\ \downarrow & & \downarrow \\ K_0(C_0(X \setminus Y, D)) & \xrightarrow{\delta_\alpha} & K_0(C_0(X, D)) \end{array} ,$$

where the vertical arrows are isomorphisms, commutes. Since  $K_1(A) = K_1(J) = 0$ , by Proposition 2.26, we get

$$\begin{aligned} K_0(C^*(A, \alpha; J)) &\cong \text{coker}(K_0(\iota) - K_0(\alpha|_J)) \cong K_0(X, \varphi, \{\alpha_x\}_{x \in \Delta}; Y), \\ K_1(C^*(A, \alpha; J)) &\cong \ker(K_0(\iota) - K_0(\alpha|_J)) \cong K_1(X, \varphi, \{\alpha_x\}_{x \in \Delta}; Y). \end{aligned}$$

□

**Remark 5.14.** We note that neither Definition 5.12 nor the proof of Theorem 5.13 makes use of the assumption that  $D$  is simple.

We are ready to give formulas for  $K$ -theory of all gauge-invariant ideals and corresponding quotients in  $C^*(C_0(X, D), \alpha; J)$ .

**Theorem 5.15.** *If  $\varphi$  is free then all ideals in  $C^*(C_0(X, D), \alpha; J)$  are gauge-invariant. In general, if  $\mathcal{I}$  is a gauge-invariant ideal in  $C^*(C_0(X, D), \alpha; J)$  and  $(V, V')$  is the corresponding  $Y$ -pair for  $(X, \varphi)$ , as described in Proposition 5.6, then*

$$(43) \quad K_*(C^*(C_0(X, D), \alpha; J)/\mathcal{I}) \cong K_*(V, \varphi|_{\Delta \cap V}, \{\alpha_x\}_{x \in \Delta \cap V}; V')$$

and

$$(44) \quad K_*(\mathcal{I}) \cong K_*(U, \varphi^Y|_{\Delta \setminus (\varphi^Y)^{-1}(U)}, \{\alpha_x\}_{x \in \Delta^Y \setminus (\varphi^Y)^{-1}(U)}),$$

where  $(X^Y, \varphi^Y)$  is the system described in Definition 5.4 and  $U := X^Y \setminus (V, V')^Y$  where  $(V, V')^Y$  is given by (39). If  $V' = V \cap Y$ , then

$$(45) \quad K_*(\mathcal{I}) \cong K_*(X \setminus V, \varphi|_{\Delta \setminus \varphi^{-1}(V)}, \{\alpha_x\}_{x \in \Delta \setminus \varphi^{-1}(V)}; X \setminus (V \cup Y)).$$

*Proof.* We get (43) by combining isomorphisms (41) and (42). Since  $\mathcal{I}$  is Morita-Rieffel equivalent to  $C^*(C_0(U, D), \alpha^J|_{C_0(U, D)})$  see Proposition 5.6, we get (44) by Theorem 5.13 and [24, Proposition B.5]. Similarly, in view of the last part of Proposition 5.6, we obtain (45). □

**Corollary 5.16.** *Suppose that  $\varphi$  is free and  $\varphi(\Delta)$  is open in  $X$ . For any ideal  $\mathcal{I}$  in  $C^*(C_0(X, D), \alpha)$  we have*

$$K_*(C^*(C_0(X, D), \alpha)/\mathcal{I}) \cong K_*(V, \varphi|_{\Delta \cap V}, \{\alpha_x\}_{x \in \Delta \cap V})$$

and

$$K_*(\mathcal{I}) \cong K_*(X \setminus V, \varphi|_{\Delta \setminus \varphi^{-1}(V)}, \{\alpha_x\}_{x \in \Delta \setminus \varphi^{-1}(V)})$$

where  $I_V = C_0(X, D) \cap \mathcal{I}$ .

*Proof.* In the proof of Theorem 5.15 apply Corollary 5.7 instead of Proposition 5.6.  $\square$

We illustrate the above results by indicating how our construction can be used to produce non-simple classifiable  $C^*$ -algebras from simple ones, only by adding an appropriate ‘dynamical ingredient’. Starting from an arbitrary Kirchberg algebra we construct a classifiable  $C^*$ -algebra with a non-Hausdorff primitive ideal space with two points. Such algebras were the first to be considered in classification of non-simple infinite  $C^*$ -algebras [50], [7].

**Example 5.17.** Let  $D$  be a Kirchberg algebra that satisfies the UCT and let  $X := \mathcal{C} \cup \{x_0\}$  be a disjoint sum of the Cantor set  $\mathcal{C}$  and a clopen singleton  $\{x_0\}$ . Then  $A = C(X, D)$  is a nuclear, separable  $C^*$ -algebra satisfying the UCT. Suppose that  $\varphi : X \rightarrow X$  is such that  $\varphi(X) = \mathcal{C}$  and  $\varphi : \mathcal{C} \rightarrow \mathcal{C}$  is a minimal homeomorphism. Then  $\mathcal{C}$  is the only non-trivial closed set invariant in  $(X, \varphi)$ . Hence  $C^*(A, \alpha)$  has the only one non-trivial ideal  $\mathcal{I}$  (in particular,  $\text{Prim}(C^*(A, \alpha))$  has two elements and is non-Hausdorff). Note that  $I_{\mathcal{C}} \cong D$  and  $\alpha_{I_{\mathcal{C}}} = 0$ . Hence  $C^*(I_{\mathcal{C}}, \alpha_{I_{\mathcal{C}}}) \cong D$ . By Theorem 2.19, we conclude that  $\mathcal{I}$  is Morita-Rieffel equivalent to  $D$ , and thus  $K_*(\mathcal{I}) = K_*(D)$ . The latter fact can be deduced using our formulas for  $K$ -theory: since  $X \setminus \mathcal{C} = \{x_0\}$  and  $\varphi|_{X \setminus \varphi^{-1}(\mathcal{C})} = \varphi|_{\emptyset}$  is the empty map, the domain of the corresponding group homomorphism  $\delta_{\alpha|_{I_{\mathcal{C}}}}^{\{x_0\}}$  is the zero group  $C_0(\emptyset, G) = \{0\}$  and the codomain is  $C_0(\{x_0\}, G) \cong G$ . Moreover, both  $C^*(A, \alpha)$  and  $C^*(A, \alpha)/\mathcal{I} \cong C^*(I_{\mathcal{C}}, \alpha_{I_{\mathcal{C}}})$  satisfy the UCT, see Proposition 2.10. Note that  $\mathcal{I}$  is not a complemented ideal in  $A$  and hence has no unit. Concluding, cf. Proposition 5.8, we get

$C^*(A, \alpha)$  is a strongly purely infinite, nuclear and separable  $C^*$ -algebra with only one non-trivial ideal  $\mathcal{I}$ , which is a unique non-unital Kirchberg algebra, satisfying the UCT, with the same  $K$ -theory as  $D$ . The non-trivial quotient  $C^*(A, \alpha)/\mathcal{I}$  is stably isomorphic to the unique stable Kirchberg algebra satisfying the UCT with the  $K$ -theory equal to  $K_*(\mathcal{C}, \varphi|_{\mathcal{C}}, \{\alpha_x\}_{x \in \mathcal{C}})$ .

Let us comment on the groups  $K_*(C^*(A, \alpha)/\mathcal{I}) \cong K_*(\mathcal{C}, \varphi|_{\mathcal{C}}, \{\alpha_x\}_{x \in \mathcal{C}})$ . In the case  $G = \mathbb{Z}$  (that is, for instance, when  $D = \mathcal{O}_{\infty}$ ) and  $K_0(\alpha_x) = id$  for every  $x \in \mathcal{C}$ , these groups coincide with  $K$ -groups of crossed products studied by Putnam in [48]. In particular, by [17, Theorem 6.2],  $K_0(C^*(A, \alpha)/\mathcal{I})$  might be any group which can be equipped with a structure of a simple dimension group. This indicates that allowing the endomorphisms  $K_0(\alpha_x)$  to be non-trivial or  $G$  not to be equal to  $\mathbb{Z}$ , we have a lot of flexibility for constructing systems with different  $K_0(C^*(A, \alpha)/\mathcal{I})$ .

Turning to  $K_1(C^*(A, \alpha)/\mathcal{I}) \cong K_1(\mathcal{C}, \varphi|_{\mathcal{C}}, \{\alpha_x\}_{x \in \mathcal{C}})$ , we note that  $f \in K_1(\mathcal{C}, \varphi|_{\mathcal{C}}, \{\alpha_x\}_{x \in \mathcal{C}})$  if and only if

$$(46) \quad f(x) = K_0(\alpha_x)(f(\varphi(x))), \quad \text{for every } x \in \mathcal{C}.$$

Thus by minimality of  $\varphi : \mathcal{C} \rightarrow \mathcal{C}$ , we see that  $f \in K_1(\mathcal{C}, \varphi|_{\mathcal{C}}, \{\alpha_x\}_{x \in \mathcal{C}})$  is uniquely determined by its value in a fixed point ( $0 \in \mathcal{C}$ , for instance). Therefore we have

$$K_1(C^*(A, \alpha)/\mathcal{I}) \cong \{g \in G : \text{there is } f \in C_0(\mathcal{C}, G) \text{ such that } f(0) = g \text{ and (46) holds}\}.$$

If  $K_0(\alpha_x) = id$  for every  $x \in \mathcal{C}$ , then  $K_1(C^*(A, \alpha)/\mathcal{I}) \cong G$ . If  $K_0(\alpha_x) = 0$  for at least one point  $x \in \mathcal{C}$ , then  $K_1(C^*(A, \alpha)/\mathcal{I}) = \{0\}$  (by Lemma 5.11 and minimality of the system). For the particular case when  $G = \mathbb{Z}$ , we have  $K_0(\alpha_x)(\tau) = m_x \cdot \tau$  for all  $\tau \in \mathbb{Z}$ , where  $m_x \in \mathbb{Z}$  is fixed for every  $x \in \mathcal{C}$ . If at least one of the numbers  $m_x$  is different than  $\pm 1$  then  $K_1(C^*(A, \alpha)/\mathcal{I}) = \{0\}$ . Indeed, then there is a non-empty open set  $U \subseteq \mathcal{C}$  and  $m \in \mathbb{Z} \setminus \{\pm 1\}$

such that  $K_0(\alpha_x)(\tau) = m \cdot \tau$  for all  $x$  in  $U$ . We may assume that  $m \neq 0$ . By minimality, the orbit of 0 visits  $U$  infinitely many times. Thus for any  $f \in K_1(\mathcal{C}, \varphi|_{\mathcal{C}}, \{\alpha_x\}_{x \in \mathcal{C}})$  the integer  $g := f(0)$  is divisible by any power of  $m$ . This implies that  $g = 0$ .

Finally, we include a simple example showing explicitly the dependence of  $K_0(C^*(A, \alpha))$  on the choice of endomorphisms  $\alpha_x$ ,  $x \in X$ .

**Example 5.18** (*K-theory for finite minimal systems*). Let  $(X, \varphi)$  be given by the relations:  $X = \{1, 2, \dots, n\}$ ,  $n > 1$ ,  $\Delta = X \setminus \{1\}$ ,  $\varphi(i) = i - 1$ , for  $i = 2, \dots, n$ . Let  $Y = X \setminus \varphi(\Delta) = \{n\}$ . Assume also that  $G = \mathbb{Z}$ . Then there are integers  $m_2, \dots, m_n$  such that  $K_0(\alpha_i)(k) = m_i k$ , for all  $i = 2, \dots, n$  and  $k \in \mathbb{Z}$ . In particular, identifying  $C_0(X, \mathbb{Z})$  and  $C_0(X \setminus Y, \mathbb{Z})$  with  $\mathbb{Z}^n$  and  $\mathbb{Z}^{n-1}$  respectively, we see that  $\delta_\alpha^Y : \mathbb{Z}^{n-1} \rightarrow \mathbb{Z}^n$  is given by the formula

$$\delta_\alpha^Y(k_1, \dots, k_{n-1}) = (0, k_2 - m_2 k_1, \dots, k_{n-1} - m_{n-1} k_{n-2}, -m_n k_{n-1}).$$

A moment of thought yields that

$$K_1(C^*(A, \alpha)) \cong \ker(\delta_\alpha^Y) \cong \begin{cases} \mathbb{Z} & \text{if } m_i = 0 \text{ for some } i, \\ \{0\} & \text{otherwise.} \end{cases}$$

The reader may check that  $\text{coker}(\delta_\alpha^Y) \cong \mathbb{Z}^2$  if  $m_i = 0$  for some  $i$ , and if the numbers  $m_i$  are non-zero then the map  $(g_1, \dots, g_n) \mapsto (g_1, \sum_{i=2}^n (\prod_{j=i+1}^n m_j^n) g_i)$  factors through to the isomorphism  $\text{coker}(\delta_\alpha^Y) \cong \mathbb{Z} \oplus \mathbb{Z}/(m_2 m_3 \dots m_n) \mathbb{Z}$ . Thus we get

$$K_0(C^*(A, \alpha)) \cong \mathbb{Z} \oplus \mathbb{Z}/(m_2 m_3 \dots m_n) \mathbb{Z}.$$

## APPENDIX A. RELATIVE CUNTZ-PIMSNER ALGEBRAS

A  $C^*$ -correspondence over a  $C^*$ -algebra  $A$  is a right Hilbert  $A$ -module  $E$  with a left action  $\phi_E : A \rightarrow \mathcal{L}(E)$  of  $A$  on  $E$  via adjointable operators. We let  $J(E) := \phi_E^{-1}(\mathcal{K}(E))$  to be the ideal in  $A$  consisting of elements that act from the left on  $E$  as generalized compact operators. For any ideal  $J$  in  $J(E)$  the relative Cuntz-Pimsner algebra  $\mathcal{O}(J, E)$  is constructed as a quotient of the  $C^*$ -algebra generated by Fock representation of  $E$ , see [42, Definition 2.18] or [37, Definition 4.9]. The  $C^*$ -algebra  $\mathcal{O}(J, E)$  is universal with respect to appropriately defined representations of  $E$ , see [15, Remark 1.4] or [37, Proposition 4.10]. Namely, a representation  $(\pi, \pi_E)$  of a  $C^*$ -correspondence  $E$  consists of a representation  $\pi : A \rightarrow \mathcal{B}(H)$  in a Hilbert space  $H$  and a linear map  $\pi_E : E \rightarrow \mathcal{B}(H)$  such that

$$\pi_E(ax \cdot b) = \pi(a)\pi_E(x)\pi(b), \quad \pi_E(x)^* \pi_E(y) = \pi(\langle x, y \rangle_A), \quad a, b \in A, x \in E.$$

Then  $\pi_E$  is automatically bounded. If  $\pi$  is faithful, then  $\pi_E$  is isometric and we say that  $(\pi, \pi_E)$  is *injective*. The  $C^*$ -subalgebra  $\mathcal{K}(E) \subseteq \mathcal{L}(E)$  of *generalized compact operators* is the closed linear span of the operators  $\Theta_{x,y}$  where  $\Theta_{x,y}(z) = x\langle y, z \rangle_A$  for  $x, y, z \in E$ . Any representation  $(\pi, \pi_E)$  of  $E$  induces a homomorphism  $(\pi, \pi_E)^{(1)} : \mathcal{K}(E) \rightarrow \mathcal{B}(H)$  which satisfies

$$(\pi, \pi_E)^{(1)}(\Theta_{x,y}) = \pi_E(x)\pi_E(y)^*, \quad (\pi, \pi_E)^{(1)}(T)\pi_E(x) = \pi_E(Tx)$$

for  $x, y \in E$  and  $T \in \mathcal{K}(E)$ . The set

$$I_{(\pi, \pi_E)} := \{a \in J(E) : (\pi, \pi_E)^{(1)}(\phi_E(a)) = \pi(a)\}$$



is an ideal in  $J(E)$ . We call  $I_{(\pi, \pi_E)}$  the *ideal of covariance* for  $(\pi, \pi_E)$ . For any ideal  $J$  contained in  $J(E)$  a representation  $(\pi, \pi_E)$  of  $E$  is said to be *J-covariant* if  $J \subseteq I_{(\pi, \pi_E)}$ . Note that if  $(\pi, \pi_E)$  is injective, then we have

$$I_{(\pi, \pi_E)} = \{a \in A : \pi(a) \in (\pi, \pi_E)^{(1)}(\mathcal{K}(E))\} \subseteq (\ker \phi_E)^\perp$$

cf. [24, Page 143].

**Proposition A.1.** *Let  $E$  be a  $C^*$ -correspondence over  $A$  and let  $J$  be an ideal in  $J(E)$ . Then there is a  $J$ -covariant representation  $(\iota, \iota_E)$  of  $E$  such that*

- i)  $\mathcal{O}(J, E)$  is generated as a  $C^*$ -algebra by  $\iota(A) \cup \iota_E(E)$ ,
- ii) for any  $J$ -covariant representation  $(\pi, \pi_E)$  of  $E$  there is a homomorphism  $\pi \rtimes_J \pi_E$  of  $\mathcal{O}(J, E)$  such that  $(\pi \rtimes_J \pi_E) \circ \iota = \pi$  and  $(\pi \rtimes_J \pi_E) \circ \iota_E = \pi_E$ .

Moreover, the representation  $(\iota, \iota_E)$  is injective if and only if  $J \subseteq (\ker \phi_E)^\perp$ .

*Proof.* The first part of the assertion is [15, Proposition 1.3]. The second part follows from [42, Proposition 2.21] and [24, Proposition 3.3].  $\square$

It follows from the above proposition that  $\mathcal{O}(J, E)$  is equipped with a gauge circle action which acts as identity on the image of  $A$  in  $\mathcal{O}(J, E)$ . We say that a representation  $(\pi, \pi_E)$  of  $E$  admits a *gauge action* if there exists a group homomorphism  $\beta : \mathbb{T} \rightarrow \text{Aut}(C^*(\pi(A) \cup \pi_E))$  such that  $\beta_z(\pi(a)) = \pi(a)$  and  $\beta_z(\pi_E(x)) = z\pi_E(x)$  for all  $a \in A$ ,  $x \in E$  and  $z \in \mathbb{T}$ .

**Proposition A.2** (Corollary 11.8 in [25]). *Let us assume that  $J$  is an ideal  $J(E) \cap (\ker \phi_E)^\perp$ . For any injective  $J$ -covariant representation  $(\pi, \pi_E)$  the homomorphism  $\pi \rtimes_J \pi_E$  of  $\mathcal{O}(J, E)$  is injective if and only if  $I_{(\pi, \pi_E)} = J$  and  $(\pi, \pi_E)$  admits a gauge action.*

Katsura, in [25], described ideals in  $\mathcal{O}(J, E)$  that are invariant under the gauge action in the following way. For any ideal  $I$  in  $A$  we define two other ideals

$$\begin{aligned} E(I) &:= \overline{\text{span}}\{\langle x, a \cdot y \rangle_A \in A : a \in I, x, y \in E\}, \\ E^{-1}(I) &:= \{a \in A : \langle x, a \cdot y \rangle_A \in I \text{ for all } x, y \in E\}. \end{aligned}$$

If  $E(I) \subseteq I$ , then the ideal  $I$  is said to be *positively invariant*, [25, Definition 4.8]. For any positively invariant ideal  $I$  we have a naturally defined quotient  $C^*$ -correspondence  $E_I = E/EI$  over  $A/I$ . Denoting by  $q_I : A \rightarrow A/I$  the quotient map one puts

$$J_E(I) := \{a \in A : \phi_{E_I}(q_I(a)) \in \mathcal{K}(E_I), aE^{-1}(I) \subseteq I\}.$$

**Definition A.3** (Definition 5.6 in [25]). Let  $E$  be a  $C^*$ -correspondence over a  $C^*$ -algebra  $A$ . A *T-pair* of  $E$  is a pair  $(I, I')$  of ideals  $I, I'$  of  $A$  such that  $I$  is positively invariant and  $I \subseteq I' \subseteq J_E(I)$ .

Exploiting the results of [25] we get the following theorem.

**Theorem A.4.** *Let  $E$  be a  $C^*$ -correspondence over a  $C^*$ -algebra  $A$  and  $J$  be an ideal of  $A$  contained in  $(\ker \phi_E)^\perp \cap J(E)$ . Then relations*

$$(47) \quad I = A \cap \mathcal{I}, \quad I' = A \cap (\mathcal{I} + EE^*)$$

*establish a bijective correspondence between T-pairs  $(I, I')$  for  $E$  with  $J \subseteq I'$  and gauge-invariant ideals  $\mathcal{I}$  in  $\mathcal{O}(J, E)$ . Moreover, for objects satisfying (47) we have*

$$\mathcal{O}(J, E)/\mathcal{I} \cong \mathcal{O}(q_I(I'), E_I),$$

and if  $\mathcal{I}$  is generated (as an ideal) by  $I$  then  $\mathcal{I}$  is Morita-Rieffel equivalent to  $\mathcal{O}(J \cap I, IE)$ .

*Proof.* The first part of the assertion follows from [25, Proposition 11.9]. Now, let  $(I, I')$  and  $\mathcal{I}$  be the corresponding objects satisfying (47) and let  $q : \mathcal{O}(J, E) \rightarrow \mathcal{O}(J, E)/\mathcal{I}$  be the quotient map. Put

$$\pi(a + I) := q(a), \quad \pi_{E_I}(x + IE) := q(x), \quad a \in A, x \in E.$$

Since  $I = A \cap \mathcal{I}$ , this yields a well defined representation  $(\pi, \pi_{E_I})$  of  $(I, E_I)$ . Since  $I' \subseteq (\mathcal{I} + EE^*)$  we have  $q_I(I') \subseteq I_{(\pi, \pi_{E_I})}$ . Thus  $(\pi, \pi_{E_I})$  gives rise to a surjection  $\mathcal{O}(q_I(I'), E_I) \rightarrow \mathcal{O}(J, E)/\mathcal{I}$ . To see it is an isomorphism note that  $(\pi, \pi_{E_I})$  admits a gauge action, because  $\mathcal{I}$  is gauge-invariant. Moreover,  $I' = A \cap (\mathcal{I} + EE^*)$  implies that  $a \in I'$  if and only if  $a + \mathcal{I} \in \mathcal{I} + EE^*$ , for any  $a \in A$ . Thus we get

$$\begin{aligned} \{q_I(a) \in A/I : \pi(q_I(a)) \in (\pi, \pi_E)^{(1)}(\mathcal{K}(E_I))\} &= \{a + I \in A/I : q(a) \in q(EE^*)\} \\ &= \{a + I \in A/I : a \in I'\} = q_I(I'). \end{aligned}$$

Hence by [25, Corollary 11.8] we get  $\mathcal{O}(q_I(I'), E_I) \cong \mathcal{O}(J, E)/\mathcal{I}$ .

Suppose now that  $\mathcal{I}$  is generated (as an ideal) by  $I$ . The embeddings of  $I$  and  $IE$  into  $\mathcal{O}(J, E)$  give rise to a faithful representation  $(\pi, \pi_{IE})$  of  $(I, IE)$  in  $\mathcal{O}(J, E)$ . Clearly,  $(\pi, \pi_{IE})$  admits a gauge action and we have

$$\begin{aligned} I_{(\pi, \pi_{IE})} &= \{a \in I : a \in (IE)(IE)^*\} = \{a \in I : a \in EE^*\} \\ &= \{a \in A : a \in EE^*\} \cap I = J \cap I. \end{aligned}$$

Hence by [25, Corollary 11.8] we see that the  $C^*$ -subalgebra  $B$  of  $\mathcal{O}(J, E)$  generated by  $I$  and  $IE$  is isomorphic to  $\mathcal{O}(J \cap I, IE)$ . It is not difficult to see, cf. the proof of [25, Proposition 9.3], that  $B = I\mathcal{O}(J, E)I$  is the hereditary subalgebra of  $\mathcal{O}(J, E)$  generated by  $I$ . Hence  $B \cong \mathcal{O}(J \cap I, IE)$  is Morita-Rieffel equivalent to the ideal  $\mathcal{I}$  generated by  $I$ .  $\square$

We recall Katsura's version of the Pimsner-Voiculescu exact sequence for a  $C^*$ -correspondence  $E$ . We consider the linking algebra  $D_E = \mathcal{K}(E \oplus A)$  in the following matrix representation

$$D_E = \begin{pmatrix} \mathcal{K}(E) & E \\ \tilde{E} & A \end{pmatrix},$$

where  $\tilde{E} = \mathcal{K}(E, A)$  is the dual Hilbert bimodule of  $E \cong \mathcal{K}(A, E)$ . Let  $\iota : J \rightarrow A$ ,  $\iota_{11} : \mathcal{K}(E) \rightarrow D_E$  and  $\iota_{22} : A \rightarrow D_E$  be inclusion maps;  $\iota_{11}(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\iota_{22}(a) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ . By [24, Proposition B.3],  $K_i(\iota_{22}) : K_i(A) \rightarrow K_i(D_E)$ ,  $i = 0, 1$ , are isomorphisms.

**Theorem A.5** (Theorem 8.6 in [24]). *Within the above notation, the following sequence is exact:*

$$(48) \quad \begin{array}{ccccc} K_0(J) & \xrightarrow{K_0(\iota) - K_0(\iota_{22})^{-1} \circ K_0(\iota_{11} \circ \phi_E|_J)} & K_0(A) & \xrightarrow{K_0(i_A)} & K_0(\mathcal{O}(J, E)) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}(J, E)) & \xleftarrow{K_1(i_A)} & K_1(A) & \xleftarrow{K_1(\iota) - K_1(\iota_{22})^{-1} \circ K_1(\iota_{11} \circ \phi_E|_J)} & K_1(J) \end{array}$$

By a *Hilbert  $A$ - $B$  bimodule* we mean  $E$  which is both a left Hilbert  $A$ -module and a right Hilbert  $B$ -module with respective inner products  $\langle \cdot, \cdot \rangle_B$  and  ${}_A\langle \cdot, \cdot \rangle$  satisfying the condition:  $x \cdot \langle y, z \rangle_B = {}_A\langle x, y \rangle \cdot z$ , for all  $x, y, z \in E$ . If additionally,  $B = \langle B, B \rangle_B$  and  $A = {}_A\langle A, A \rangle$ , we say that  $A$  and  $B$  are *Morita-Rieffel equivalent* (or *strongly Morita equivalent*). A Hilbert  $A$ - $A$  bimodule is also called a *Hilbert bimodule over  $A$* . If  $E$  is a Hilbert bimodule over  $A$  then it is also a  $C^*$ -correspondence and Katsura's algebra  $\mathcal{O}_E$  associated to  $E$  coincides with the  $C^*$ -algebra associated to  $E$  in [3], see [23, Proposition 3.7].

Suppose that  $E$  is a Hilbert bimodule over  $A$ . Then  $E$  induces a *partial homeomorphism*  $\widehat{E}$  of  $\widehat{A}$  dual to  $E$ , see [32, Definition 1.1]. More specifically,  $\langle E, E \rangle_A$  and  ${}_A\langle E, E \rangle$  are ideals in  $A$  and  $\widehat{E} : \widehat{\langle E, E \rangle_A} \rightarrow \widehat{{}_A\langle E, E \rangle}$  is a homeomorphism, which factors through the induced representation functor  $E$ -Ind. The latter is defined as follows: if  $\pi : A \rightarrow \mathcal{B}(H)$  is a representation, then  $E$ -Ind( $\pi$ ) :  $A \rightarrow \mathcal{B}(E \otimes_\pi H)$  is a representation where the Hilbert space  $E \otimes_\pi H$  is generated by simple tensors  $x \otimes_\pi h$ ,  $x \in E$ ,  $h \in H$ , satisfying  $\langle x_1 \otimes_\pi h_1, x_2 \otimes_\pi h_2 \rangle = \langle h_1, \pi(\langle x_1, x_2 \rangle_A) h_2 \rangle$ , and

$$E\text{-Ind}(\pi)(a)(x \otimes_\pi h) = (ax) \otimes_\pi h, \quad a \in A.$$

By [32, Theorems 2.2 and 2.5] we have the following result.

**Theorem A.6.** *Let  $\widehat{E}$  be the partial homeomorphism of  $\widehat{A}$  associated to a Hilbert bimodule  $E$ . If  $\widehat{E}$  is topologically free, then every non-zero ideal in  $\mathcal{O}_E$  has a non-zero intersection of  $A$ . If  $\widehat{E}$  is free, then every ideal in  $\mathcal{O}_E$  is gauge-invariant.*

**$C^*$ -correspondences associated to  $C^*$ -dynamical systems.** Let us now fix  $C^*$ -dynamical system  $(A, \alpha)$ . We associate to  $(A, \alpha)$  the  $C^*$ -correspondence given by

$$E_\alpha := \alpha(A)A, \quad a \cdot x := \alpha(a)x, \quad x \cdot a := xa, \quad \langle x, y \rangle_A := x^*y,$$

where  $a \in A$ ,  $x, y \in E$ . Clearly, we have  $\ker \phi_{E_\alpha} = \ker \alpha$ .

**Lemma A.7.** *We have  $J(E_\alpha) = A$  and the map  $\mathcal{K}(E_\alpha) \ni \Theta_{x,y} \mapsto xy^* \in \alpha(A)A\alpha(A)$  yields an isomorphism of  $C^*$ -algebras.*

*Proof.* The proof is straightforward and thus left to the reader.  $\square$

**Proposition A.8.** *For any ideal  $J$  in  $(\ker \alpha)^\perp$  there is a natural isomorphism  $C^*(A, \alpha; J) \cong \mathcal{O}(J, E_\alpha)$ . More precisely, the relation*

$$\pi_{E_\alpha}(x) = U^*\pi(x), \quad x \in \alpha(A)A,$$

*yields a one-to-one correspondence between representations  $(\pi, U)$  of  $(A, \alpha)$  and representations  $(\pi, \pi_{E_\alpha})$  of  $E_\alpha$  where  $\pi : A \rightarrow \mathcal{B}(H)$  is a non-degenerate representation. For the corresponding representations we have  $I_{(\pi, \pi_{E_\alpha})} = I_{(\pi, U)}$ .*

*Proof.* By [35, Proposition 3.26] crossed product by  $\alpha$  treated as a completely positive map coincides with the crossed product considered in the present paper (note that an operator  $S$  in [35] plays the role of  $U^*$ ). By [35, Lemma 3.25] the GNS  $C^*$ -correspondence associated to  $\alpha$  (treated as a completely positive map) is naturally isomorphic to  $E_\alpha$ . Thus the assertion follows from [35, Propositions 3.10].  $\square$

Some of the following facts were stated without proof in [33, Appendix A].

**Proposition A.9.** *An ideal  $I$  in  $A$  is positively invariant for  $E_\alpha$  if and only if  $I$  is positively invariant in  $(A, \alpha)$ . Moreover, if  $I$  is positively invariant, then we have natural identifications:*

$$IE_\alpha = E_{\alpha|_I}, \quad (E_\alpha)_I = E_{\alpha_I}.$$

*Proof.* Clearly,  $\alpha(I) \subseteq I$  if and only if  $E_\alpha(I) = A\alpha(I)A \subseteq I$ , which proves the first part of the assertion. If  $I$  is positively invariant then  $\alpha(I)A = \alpha(I)IA = \alpha(I)I$ , which allows us the identification  $IE_\alpha = E_{\alpha|_I}$ . The natural algebraic isomorphism  $E_{\alpha_I} = \alpha_I(A/I)A/I = q_I(\alpha(A)A) \cong \alpha(A)A/\alpha(A)I = (E_\alpha)_I$  intertwines the operations of  $C^*$ -correspondences. Hence it is an isomorphism that allows us the identification  $(E_\alpha)_I = E_{\alpha_I}$ .  $\square$

**Proposition A.10.** *Suppose that  $I$  and  $I'$  are ideals in  $A$ .*

$$(I, I') \text{ is a } T\text{-pair for } E_\alpha \text{ with } J \subseteq I' \iff (I, I') \text{ is a } J\text{-pair for } (A, \alpha).$$

*Proof.* We have  $E_\alpha^{-1}(I) = \{a \in A : x^*\alpha(a)y \in I \text{ for all } x, y \in \alpha(A)A\} = \alpha^{-1}(I)$ . By Proposition A.9 we may identify  $(E_\alpha)_I$  with  $E_{\alpha_I}$ . Hence  $J((E_\alpha)_I) = A/I$  and we get  $J_{E_\alpha}(I) = \{a \in A : a\alpha^{-1}(I) \subseteq I\}$ . Thus if we assume that  $I$  is positively invariant, cf. Proposition A.9, then we get

$$I \subseteq I' \subseteq J_{E_\alpha}(I) \iff I' \cap \alpha^{-1}(I) = I.$$

This implies the assertion.  $\square$

We recall, see [23, Subsection 3.3] or [31, Proposition 1.11], that a  $C^*$ -correspondence  $E$  over  $A$  is a Hilbert bimodule over  $A$  if and only if  $\phi_E : (\ker \phi_E)^\perp \cap J(E) \rightarrow \mathcal{K}(E)$  is an isomorphism, and then  ${}_A\langle x, y \rangle = \phi_E^{-1}(\Theta_{x,y})$ .

**Proposition A.11.** *The  $C^*$ -correspondence  $E_\alpha$  is a Hilbert bimodule over  $A$  if and only if  $(A, \alpha)$  is reversible and then*

$${}_A\langle x, y \rangle = \alpha^{-1}(xy^*)$$

where  $\alpha^{-1}$  is the inverse to the isomorphism  $\alpha : (\ker \alpha)^\perp \rightarrow \alpha(A)$ .

*Proof.* Clearly, we have  $(\ker \phi_{E_\alpha})^\perp = (\ker \alpha)^\perp$ . Thus Lemma A.7 implies that  $\phi_{E_\alpha} : (\ker \phi_{E_\alpha})^\perp \cap J(E_\alpha) = (\ker \alpha)^\perp \rightarrow \mathcal{K}(E) \cong \alpha(A)A\alpha(A)$  is an isomorphism if and only if the system  $(A, \alpha)$  is reversible.  $\square$

Let us now consider a reversible  $C^*$ -dynamical system  $(A, \alpha)$  and the corresponding Hilbert bimodule  $E_\alpha$ . Clearly,  ${}_A\langle E_\alpha, E_\alpha \rangle = (\ker \alpha)^\perp$  and  $\langle E_\alpha, E_\alpha \rangle_A = A\alpha(A)A$ . Under the standard identifications we have  $\widehat{\alpha(A)} = \{[\pi] \in \widehat{A} : \pi(\alpha(A)) \neq 0\} = \widehat{\langle E, E \rangle}_A$ . The partial homeomorphism dual to  $E_\alpha$  can be identified with the one described in Definition 2.34:

**Lemma A.12.** *Let  $(A, \alpha)$  be a reversible  $C^*$ -dynamical system. The homeomorphisms  $\widehat{\alpha(A)} : \widehat{\alpha(A)} \rightarrow (\ker \alpha)^\perp$  and  $\widehat{E_\alpha} : \widehat{\langle E_\alpha, E_\alpha \rangle}_A \rightarrow {}_A\widehat{\langle E_\alpha, E_\alpha \rangle}$  coincide.*

*Proof.* Let  $\pi : A \rightarrow \mathcal{B}(H)$  be an irreducible representation such that  $\pi(\alpha(A)) \neq 0$ . Then  $\widehat{\alpha}([\pi])$  is the equivalence class of the representation  $\pi \circ \alpha : A \rightarrow \mathcal{B}(\pi(\alpha(A))H)$ . Since  $\pi(\alpha(A))H = \pi(\alpha(A)A)H$  and  $\|\sum_i a_i \otimes_\pi h_i\|^2 = \|\sum_{i,j} \langle h_i, \pi(a_i^* a_j) h_j \rangle\| = \|\sum_i \pi(a_i) h_i\|^2$ , for  $a_i \in E_\alpha = \alpha(A)A$ ,  $h_i \in H$ ,  $i = 1, \dots, n$ , we see that  $a \otimes_\pi h \mapsto \pi(a)h$  yields a unitary operator  $U : E_\alpha \otimes_\pi H \rightarrow \pi(\alpha(A))H$ . Furthermore, for  $a \in A$ ,  $b \in \alpha(A)$  and  $h \in H$  we have

$$[E_\alpha\text{-Ind}(\pi)(a)U^*]\pi(b)h = E_\alpha\text{-Ind}(\pi)(a)b \otimes_\pi h = (\alpha(a)b) \otimes_\pi h = [U^*(\pi \circ \alpha)(a)]\pi(b)h.$$

Hence  $U$  intertwines  $E_\alpha$ -Ind and  $\pi \circ \alpha$ . This proves that  $\widehat{E_\alpha} = \widehat{\alpha}$ .  $\square$

We get the exact sequence for crossed products by endomorphisms, see Proposition 2.26, by using (48) and the following lemma.

**Lemma A.13.** *Let  $J$  be an ideal in  $(\ker \alpha)^\perp$ . With the notation preceding Theorem A.5 applied to  $E_\alpha$  we have  $K_i(\iota_{22} \circ \alpha|_J) = K_i(\iota_{11} \circ \phi_{E_\alpha}|_J)$ ,  $i = 1, 2$ .*

*Proof.* For brevity, we put  $E := E_\alpha$ . Let  $\flat : E \rightarrow \tilde{E}$  be the canonical antilinear isomorphism, and let  $\alpha^{-1}$  be the inverse to the isomorphism  $\alpha : (\ker \alpha)^\perp \rightarrow \alpha((\ker \alpha)^\perp)$ . Plainly, the map

$$M_2(\alpha(J)) \ni \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \xrightarrow{\Phi} \begin{pmatrix} \phi_E(\alpha^{-1}(a_{11})) & a_{12} \\ \flat(a_{21}^*) & a_{22} \end{pmatrix} \in D_E,$$

is a homomorphism of  $C^*$ -algebras. The following diagram commutes:

$$\begin{array}{ccc} & J & \\ \iota_{11} \circ \alpha \swarrow & & \searrow \iota_{11} \circ \phi_E \\ M_2(\alpha(J)) & \xrightarrow{\Phi} & D_E \end{array}.$$

Therefore

$$K_i(\iota_{11} \circ \phi_E|_J) = K_i(\Phi \circ \iota_{11} \circ \alpha|_J), \quad i = 0, 1.$$

Recall that for any  $C^*$ -algebra  $B$  the homomorphisms  $\iota_{ii} : B \rightarrow M_2(B)$ ,  $i = 1, 2$ , induce the same mappings on  $K$ -groups. Thus  $K_i(\iota_{11} \circ \alpha|_J) = K_i(\iota_{22} \circ \alpha|_J)$ ,  $i = 1, 2$ . By the form of  $\Phi$  we see that  $\Phi \circ \iota_{22} \circ \alpha = \iota_{22} \circ \alpha$  on  $J$ . Concluding, for  $i = 0, 1$  we get

$$K_i(\iota_{11} \circ \phi_E|_J) = K_i(\Phi \circ \iota_{11} \circ \alpha|_J) = K_i(\Phi \circ \iota_{22} \circ \alpha|_J) = K_i(\iota_{22} \circ \alpha|_J).$$

$\square$

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